

ASYMPTOTIC ANALYSIS OF MULTISCALE MARKOV CHAIN

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Abstract. We consider continuous-time Markov chain on a finite state space X . We assume X can be clustered into several subsets such that the intra-transition rates within these subsets are of order $\mathcal{O}(\frac{1}{\epsilon})$ comparing to the inter-transition rates among them, where $0 < \epsilon \ll 1$. Several asymptotic results are obtained as $\epsilon \rightarrow 0$ concerning the convergence of Kolmogorov backward equation, Poincaré constant, (modified) logarithmic Sobolev constant to their counterparts of certain reduced Markov chain. Both reversible and irreversible Markov chains are considered.

Key words. multiple time scale, continuous-time Markov chain, asymptotic analysis, Poincaré constant, logarithmic Sobolev constant.

AMS subject classifications. 60J27, 34E13, 34E05

1. Introduction.

1.1. Multiscale Markov chain. In recent decades, Markov chains have been intensively investigated due to their effectiveness in modeling systems arising from biology, physics, economics et al. [17, 14, 19]. Inspired by new phenomena from these disciplines, new topics related to Markov chains are continuously emerging and attracting researchers' attentions. Metastability in Markov chains is one such interesting topic which tries to understand systems' behaviors on large time scales by eliminating systems' oscillations on short time scales and identifying certain effective dynamics on large time scales [22, 4].

In this work, we consider a continuous-time Markov chain \mathcal{C} on finite state space $X = \{x_1, x_2, \dots, x_n\}$. We assume \mathcal{C} is irreducible and therefore has a unique invariant measure [14, 19]. Suppose state space X can be clustered into m ($m > 1$) nonempty disjoint subsets X_1, X_2, \dots, X_m , with $|X_i| = n_i > 0$, $\sum_{i=1}^m n_i = n$. We will be interested in the situation when transitions of system's states within the same subset occur much more frequently than transitions between states belonging to different subsets. Precisely, let $n \times n$ matrix Q be the infinitesimal generator of Markov chain \mathcal{C} , which we assume can be written as

$$Q = \frac{1}{\epsilon} Q_0 + Q_1, \quad (1.1)$$

for some parameter $0 < \epsilon \ll 1$. Matrices Q_0 and Q_1 satisfy that

1. $Q_0(x, y) \geq 0, Q_1(x, y) \geq 0$, if $x \neq y$.
2. Each row sum of their entries equals zero, i.e.

$$\sum_{y \in X} Q_0(x, y) = \sum_{y \in X} Q_1(x, y) = 0, \quad \forall x \in X.$$

3. $Q_0(x, y) = 0$, if $x \in X_i, y \in X_j$ for some $1 \leq i \neq j \leq m$.
4. $Q_1(x, y) = 0$, if $x, y \in X_i$ for some $1 \leq i \leq m$ and $x \neq y$.

That is, Q_0 and Q_1 describe the intra- and inter-transition rates among subsets X_i , respectively. Notice that comparing to Chapter 5 and 9 of [21], our setting is more general in that each subset

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may contain different number of states and transitions between states are less restrictive. From the above assumptions, we can rearrange states in X such that

$$Q_0 = \text{diag}\{Q_{0,1}, Q_{0,2}, \dots, Q_{0,m}\} \quad (1.2)$$

is a block diagonal matrix consisting of m submatrices $Q_{0,i}$, $1 \leq i \leq m$, where $Q_{0,i}$ is an $n_i \times n_i$ matrix and defines a Markov chain \mathcal{C}_i on subset X_i . We further assume that for each $1 \leq i \leq m$, Markov chain \mathcal{C}_i is irreducible and therefore has a unique invariant measure π_i . Let π^ϵ be the unique invariant measure of Markov chain \mathcal{C} on X . Given $1 \leq i, j \leq m$, we define

$$\bar{Q}(i, j) = \sum_{x \in X_i, y \in X_j} Q_1(x, y) \pi_i(x). \quad (1.3)$$

It is direct to verify that matrix \bar{Q} in (1.3) defines an infinitesimal generator (non-negative off-diagonal elements with zero row sums) of Markov chain $\bar{\mathcal{C}}$ on $\bar{X} = \{1, 2, \dots, m\}$. We will call $\bar{\mathcal{C}}$ the reduced Markov chain and assume it has a unique invariant measure w .

The main aim of this paper is to consider several objects associated with Markov chain \mathcal{C} and their counterparts associated with Markov chain $\bar{\mathcal{C}}$. For this purpose, we first introduce the Kolmogorov backward equations

$$\frac{d}{dt} \rho_t = Q \rho_t = \left(\frac{1}{\epsilon} Q_0 + Q_1 \right) \rho_t, \quad \frac{d}{dt} \bar{\rho}_t = \bar{Q} \bar{\rho}_t, \quad (1.4)$$

where $\rho_t : X \rightarrow \mathbb{R}$ and $\bar{\rho}_t : \bar{X} \rightarrow \mathbb{R}$, $t \geq 0$. These equations play an important role in understanding the dynamical behaviors of Markov chain \mathcal{C} and $\bar{\mathcal{C}}$ [19, 21].

We will also consider constants characterizing the speed of Markov chain converging to equilibrium [9, 13, 12]. Let $\mathbf{E}_{\pi^\epsilon}$, $\text{Var}_{\pi^\epsilon}$ denote the expectation and variance with respect to measure π^ϵ respectively. First recall the definition of Poincaré constant and logarithmic Sobolev constant for Markov chain \mathcal{C} , which are defined as

$$\lambda_\epsilon = \inf_f \left\{ \frac{\mathcal{E}_\epsilon(f, f)}{\text{Var}_{\pi^\epsilon} f} \mid \text{Var}_{\pi^\epsilon} f > 0, f : X \rightarrow \mathbb{R} \right\} \quad (1.5)$$

$$\alpha_\epsilon = \inf_f \left\{ \frac{\mathcal{E}_\epsilon(f, f)}{\text{Ent}_{\pi^\epsilon}(f^2)} \mid \text{Ent}_{\pi^\epsilon}(f^2) > 0, f : X \rightarrow \mathbb{R} \right\}, \quad (1.6)$$

where the infima are taken among all non-constant functions, \mathcal{E}_ϵ , $\text{Ent}_{\pi^\epsilon}$ are the Dirichlet form and relative entropy with respect to π^ϵ , defined as

$$\mathcal{E}_\epsilon(f, g) = -\langle f, Qg \rangle_{\pi^\epsilon} = -\left\langle f, \left(\frac{Q_0}{\epsilon} + Q_1 \right) g \right\rangle_{\pi^\epsilon}, \quad f, g : X \rightarrow \mathbb{R}, \quad (1.7)$$

$$\text{Ent}_{\pi^\epsilon}(f) = \sum_{x \in X} f(x) \ln \frac{f(x)}{\mathbf{E}_{\pi^\epsilon} f} \pi^\epsilon(x), \quad f : X \rightarrow \mathbb{R}^+. \quad (1.8)$$

For function $f : X \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \mathcal{E}_\epsilon(f, f) &= \frac{1}{2\epsilon} \sum_{i=1}^m \sum_{x, x' \in X_i} (f(x') - f(x))^2 Q_{0,i}(x, x') \pi^\epsilon(x) \\ &\quad + \frac{1}{2} \sum_{i \neq j} \sum_{x \in X_i, y \in X_j} (f(y) - f(x))^2 Q_1(x, y) \pi^\epsilon(x), \end{aligned} \quad (1.9)$$

which holds in both reversible and non-reversible case [5].

The modified logarithmic Sobolev constant is defined as

$$\gamma_\epsilon = \inf_f \left\{ \frac{\mathcal{E}_\epsilon(f, \ln f)}{\text{Ent}_{\pi^\epsilon}(f)} \mid \text{Ent}_{\pi^\epsilon}(f) > 0, f : X \rightarrow \mathbb{R}^+ \right\}, \quad (1.10)$$

where the infimum is taken among all non-constant and non-negative functions. It is known that these constants satisfy

$$4\alpha_\epsilon \leq \gamma_\epsilon \leq 2\lambda_\epsilon, \quad (1.11)$$

in the reversible case, and

$$2\alpha_\epsilon \leq \gamma_\epsilon \leq 2\lambda_\epsilon, \quad 2\alpha_\epsilon \leq \lambda_\epsilon, \quad (1.12)$$

in the non-reversible case. See [2, 3, 5, 10] and references therein for more details. Let $\bar{\mathcal{E}}$ denote the Dirichlet form of the reduced Markov chain $\bar{\mathcal{C}}$. Its Poincaré constant $\bar{\lambda}$, logarithmic Sobolev constant $\bar{\alpha}$, as well as the modified logarithmic Sobolev constant $\bar{\gamma}$ can be defined similarly as in (1.5), (1.6), (1.10), by replacing \mathcal{E}_ϵ , π^ϵ , Q with $\bar{\mathcal{E}}$, w , \bar{Q} respectively. Correspondingly, they satisfy the inequality

$$4\bar{\alpha} \leq \bar{\gamma} \leq 2\bar{\lambda}, \quad (1.13)$$

in the reversible case, and

$$2\bar{\alpha} \leq \bar{\gamma} \leq 2\bar{\lambda}, \quad 2\bar{\alpha} \leq \bar{\lambda}, \quad (1.14)$$

in the non-reversible case.

Briefly speaking, in this paper we will establish the convergence of ρ_t to $\bar{\rho}_t$ in (1.4), and the convergence of constants λ_ϵ , α_ϵ , γ_ϵ in (1.5), (1.6), (1.10) to their counterpart $\bar{\lambda}$, $\bar{\alpha}$ and $\bar{\gamma}$ respectively.

1.2. Notations. In this subsection we collect some notations and definitions used in this paper. Let Ω be a finite set. For function $f : \Omega \rightarrow \mathbb{R}$,

$$|f|_\infty := \max_{x \in \Omega} |f(x)|, \quad |f|_2 := \left(\sum_{x \in \Omega} f^2(x) \right)^{\frac{1}{2}} \quad (1.15)$$

are the L^∞ norm and L^2 norm of f . Given a matrix A of order $k \times l$, denote its infinity norm as $\|A\|_\infty$, i.e. $\|A\|_\infty = \sup_{1 \leq i \leq k} \sum_{j=1}^l |a_{ij}|$. For matrix Q_1 in (1.1), we define

$$Q_\infty := \|Q_1\|_\infty = \max_{x \in X} \sum_{x' \in X} |Q_1(x, x')| = 2 \max_{x \in X} \sum_{x' \neq x} Q_1(x, x'), \quad (1.16)$$

where we have used the fact that the off-diagonal entries of Q_1 are non-negative, and $Q_1(x, x) = - \sum_{x' \neq x} Q_1(x, x') < 0$, $\forall x \in X$. From definition (1.3) of matrix \bar{Q} , it is direct to check $\|\bar{Q}\|_\infty \leq Q_\infty$.

Let μ be a probability measure over set Ω , $L^2(\mu)$ is the Hilbert space consisting of all real functions on Ω with inner product

$$\langle f, g \rangle_\mu = \sum_{x \in \Omega} f(x)g(x)\mu(x), \quad \forall f, g : \Omega \rightarrow \mathbb{R},$$

and its norm is denoted as $|\cdot|_{2,\mu}$. We write

$$\mathbf{E}_\mu f = \sum_{x \in \Omega} f(x) \mu(x), \quad \text{Var}_\mu f = \sum_{x \in \Omega} (f(x) - \mathbf{E}_\pi f)^2 \mu(x), \quad (1.17)$$

as the expectation and the variance of function f with respect to μ .

For Markov chain \mathcal{C}_i whose infinitesimal generator is $Q_{0,i}$, we denote its Dirichlet form, Poincaré constant, logarithmic Sobolev constant, modified logarithmic Sobolev constant as \mathcal{E}_i , λ_i , α_i and γ_i , respectively. Also set

$$\lambda_{\min} = \min_i \lambda_i, \quad \alpha_{\min} = \min_i \alpha_i, \quad \gamma_{\min} = \min_i \gamma_i.$$

Given function $f : X \rightarrow \mathbb{R}$ and $1 \leq i \leq m$, $f(i, \cdot)$ denotes the vector of length n_i consisting of components $f(x)$ for $x \in X_i$, while \tilde{f} denotes a function on \bar{X} , defined by $\tilde{f}(i) = \sum_{x \in X_i} f(x) \pi_i(x)$, $1 \leq i \leq m$.

We also need some notations when studying the general non-reversible case. Define

$$\begin{aligned} \Gamma &:= \text{tr}(Q_1 Q_1^T) = \sum_{x \in X} \sum_{y \in X} Q_1(y, x)^2, \\ d &:= \max_{x \in X} \left| \left\{ y \in X \mid Q_1(x, y) \neq 0 \right\} \right|, \end{aligned} \quad (1.18)$$

where $|\cdot|$ denotes the cardinality of a given set. σ_i and $\bar{\sigma}$ denote the smallest nonzero singular value of matrix $Q_{0,i}$ and \bar{Q} , respectively. Also set $\sigma_{\min} = \min_i \sigma_i$.

The paper is organized as follows. Section 2 is devoted to obtain several asymptotic results when Markov chain \mathcal{C} is reversible. The general Markov chain without reversibility assumption is studied in Section 3. In Section 4, we discuss our results and make conclusions. Appendix A collects some useful facts related to continuous-time Markov chain. Appendix B contains formal arguments which motivates our asymptotic results.

2. Asymptotic analysis : reversible case. In this section, we establish several asymptotic convergence results under the assumption that Markov chain \mathcal{C} is reversible.

2.1. Invariant measure. We start with the invariant measure π^ϵ . Taking the structure of matrix Q in (1.1), (1.2) into consideration, the detailed balance condition reads

$$\begin{aligned} \pi^\epsilon(x) Q_{0,i}(x, x') &= \pi^\epsilon(x') Q_{0,i}(x', x), \quad \text{if } x, x' \in X_i, \\ \pi^\epsilon(x) Q_1(x, y) &= \pi^\epsilon(y) Q_1(y, x), \quad \text{if } x \in X_i, y \in X_j, i \neq j. \end{aligned} \quad (2.1)$$

Since we assume Markov chain \mathcal{C}_i has a unique invariant measure, the first equation above implies that \mathcal{C}_i is also reversible, for $1 \leq i \leq m$, and $\exists w^\epsilon(i) > 0$ s.t.

$$\pi^\epsilon(x) = w^\epsilon(i) \pi_i(x), \quad \text{for } x \in X_i. \quad (2.2)$$

We have

$$\sum_{i=1}^m w^\epsilon(i) = \sum_{x \in X} \pi^\epsilon(x) = 1. \quad (2.3)$$

Substituting relation (2.2) into the second equation of (2.1) and summing up all states $x \in X_i, y \in X_j$, we obtain

$$w^\epsilon(i) \bar{Q}(i, j) = w^\epsilon(j) \bar{Q}(j, i), \quad 1 \leq i \neq j \leq m, \quad (2.4)$$

where matrix \bar{Q} is defined in (1.3). Equation (2.3) and (2.4) imply that w^ϵ coincides with the invariant measure w of Markov chain $\bar{\mathcal{C}}$ and furthermore, $\bar{\mathcal{C}}$ is reversible with respect to w . From (2.2) we also know that π^ϵ is independent of parameter ϵ . In the following of this section we will denote it as π for simplicity.

2.2. Kolmogorov backward equation. We consider the Kolmogorov backward equation

$$\frac{d}{dt}\rho_t = Q\rho_t = \left(\frac{1}{\epsilon}Q_0 + Q_1\right)\rho_t \quad (2.5)$$

with initial condition ρ_0 (ρ_0 can be negative), or more explicitly,

$$\frac{d}{dt}\rho_t(x) = \frac{1}{\epsilon} \sum_{x' \neq x, x' \in X_i} (\rho_t(x') - \rho_t(x))Q_{0,i}(x, x') + \sum_{y \notin X_i} (\rho_t(y) - \rho_t(x))Q_1(x, y), \quad (2.6)$$

for $x \in X_i$, $1 \leq i \leq m$. Multiplying both sides of (2.6) by $\pi_i(x)$, summing up states $x \in X_i$, and noticing that $Q_{0,i}^T \pi_i = 0$, we obtain the equation of $\tilde{\rho}_t(i) = \sum_{x \in X_i} \rho_t(x)\pi_i(x)$ as

$$\frac{d}{dt}\tilde{\rho}_t(i) = \sum_{x \in X_i} \sum_{y \notin X_i} (\rho_t(y) - \rho_t(x))Q_1(x, y)\pi_i(x), \quad 1 \leq i \leq m. \quad (2.7)$$

We also introduce the Kolmogorov backward equation of the reduced Markov chain $\bar{\mathcal{C}}$

$$\frac{d}{dt}\bar{\rho}_t = \bar{Q}\bar{\rho}_t = \sum_{j \neq i} (\bar{\rho}_t(j) - \bar{\rho}_t(i))\bar{Q}(i, j) \quad (2.8)$$

with initial condition $\bar{\rho}_0 = \tilde{\rho}_0$, where matrix \bar{Q} is defined in (1.3). We have

THEOREM 2.1. *Assume Markov chain \mathcal{C} is reversible. Consider functions ρ_t , $\tilde{\rho}_t$ and $\bar{\rho}_t$, which are solutions of equation (2.5), (2.7) and (2.8), respectively. For $t \geq 0$, we have*

$$|\rho_t(\cdot) - \tilde{\rho}_t(i)\mathbf{1}|_{2, \pi_i} \leq \left(e^{-\frac{\lambda_i t}{\epsilon}} + \frac{2\epsilon}{\lambda_i}Q_\infty\right)|\rho_0|_\infty, \quad (2.9)$$

$$|\tilde{\rho}_t - \bar{\rho}_t|_{2, w} \leq \frac{Q_\infty|\rho_0|_\infty}{\lambda_{\min}} \left(\min\left\{\frac{1}{\min_{i, x \in X_i} \pi_i(x)}, \frac{m}{2}\right\}\right)^{\frac{1}{2}} \left(\frac{2Q_\infty}{\lambda} + 1\right)\epsilon, \quad (2.10)$$

where constants involved are defined in Section 1.

Before entering the proof, we would like to reinterpret the results of Theorem 2.1 by considering the corresponding Markov chain processes. Let $x_t \in X$ and $\bar{x}_t \in \bar{X}$ be the Markov chain \mathcal{C} and $\bar{\mathcal{C}}$, respectively. Given function $f : X \rightarrow \mathbb{R}$ and defining $\tilde{f}(i) = \sum_{x \in X_i} f(x)\pi_i(x)$ as before, we consider quantities

$$f_t(x) = \mathbf{E}(f(x_t) \mid x_0 = x), \quad \tilde{f}_t(i) = \mathbf{E}(f(x_t) \mid x_0 \sim \pi_i), \quad x \in X, \quad (2.11)$$

then we know f_t satisfies equation (2.5) with initial condition $f_0 = f$, while $\tilde{f}_t(i)$ satisfies (2.7) with ρ_t replaced by f_t . Similarly define

$$\bar{f}_t(i) = \mathbf{E}(\tilde{f}(\bar{x}_t) \mid \bar{x}_0 = i), \quad 1 \leq i \leq m, \quad (2.12)$$

then \bar{f} satisfies (2.8) with initial condition $\bar{f}_0 = \tilde{f}_0 = \tilde{f}$. Theorem 2.1 implies

COROLLARY 2.2. Consider reversible Markov chains $x_t \in X$ and $\bar{x}_t \in \bar{X}$ defined by infinitesimal generator Q and \bar{Q} , respectively. Given $f : X \rightarrow \mathbb{R}$, define the quantities $f_t, \tilde{f}_t, \bar{f}_t$ by (2.11) and (2.12). We have $\forall t \geq 0$,

$$\begin{aligned} |f_t(i, \cdot) - \tilde{f}_t(i)\mathbf{1}|_{2, \pi_i} &\leq \left(e^{-\frac{\lambda_i t}{\epsilon}} + \frac{2\epsilon}{\lambda_i} Q_\infty\right) |f|_\infty, \quad 1 \leq i \leq m, \\ |\tilde{f}_t - \bar{f}_t|_{2, w} &\leq \frac{Q_\infty |f|_\infty}{\lambda_{\min}} \left(\min \left\{ \frac{1}{\min_{i, x \in X_i} \pi_i(x)}, \frac{m}{2} \right\} \right)^{\frac{1}{2}} \left(\frac{2Q_\infty}{\lambda} + 1 \right) \epsilon. \end{aligned} \quad (2.13)$$

Now consider a probability measure μ on X and define probability measure $\tilde{\mu}$ on \bar{X} by $\tilde{\mu}(i) = \sum_{x \in X_i} \mu(x)$, $1 \leq i \leq m$. Also define

$$\begin{aligned} \mu_t(x) &= \mathbf{P}(x_t = x | x_0 \sim \mu), \quad x \in X, \\ \tilde{\mu}_t(i) &= \mathbf{P}(x_t \in X_i | x_0 \sim \mu) = \sum_{x \in X_i} \mu_t(x), \quad 1 \leq i \leq m, \\ \bar{\mu}_t(i) &= \mathbf{P}(\bar{x}_t = i | \bar{x}_0 \sim \tilde{\mu}), \quad 1 \leq i \leq m, \end{aligned} \quad (2.14)$$

and the probability densities with respect to invariant measures π and w

$$\rho_t = \frac{d\mu_t}{d\pi}, \quad \tilde{\rho}_t = \frac{d\tilde{\mu}_t}{dw}, \quad \bar{\rho}_t = \frac{d\bar{\mu}}{dw}. \quad (2.15)$$

Recalling the detailed balance condition (2.1), we can check that functions $\rho_t, \tilde{\rho}_t$ and $\bar{\rho}_t$ satisfy equation (2.5), (2.7) and (2.8), respectively (see Appendix A). Therefore Theorem 2.1 implies

COROLLARY 2.3. Consider reversible Markov chains $x_t \in X$ and $\bar{x}_t \in \bar{X}$ defined by infinitesimal generator Q and \bar{Q} , respectively. Given probability measure μ on space X . Let $\rho_t, \tilde{\rho}_t$ and $\bar{\rho}_t$ be the density of probability measures defined by (2.14) and (2.15). For $t \geq 0$, we have

$$\begin{aligned} |\rho_t(i, \cdot) - \tilde{\rho}_t(i)\mathbf{1}|_{2, \pi_i} &\leq \left(e^{-\frac{\lambda_i t}{\epsilon}} + \frac{2\epsilon}{\lambda_i} Q_\infty\right) |\rho_0|_\infty, \quad 1 \leq i \leq m, \\ |\tilde{\rho}_t - \bar{\rho}_t|_{2, w} &\leq \frac{Q_\infty |\rho_0|_\infty}{\lambda_{\min}} \left(\min \left\{ \frac{1}{\min_{i, x \in X_i} \pi_i(x)}, \frac{m}{2} \right\} \right)^{\frac{1}{2}} \left(\frac{2Q_\infty}{\lambda} + 1 \right) \epsilon. \end{aligned} \quad (2.16)$$

Proof of Theorem 2.1 :

1. We start with the first inequality (2.9) concerning ρ_t and $\tilde{\rho}_t$. For ρ_t satisfying (2.5), we know $|\rho_t|_\infty \leq |\rho_0|_\infty$, $t \geq 0$ (can be easily seen from (2.11)). For the right hand side of (2.7), we have

$$\begin{aligned} &\left| \sum_{x \in X_i} \sum_{y \notin X_i} (\rho_t(y) - \rho_t(x)) Q_1(x, y) \pi_i(x) \right| \\ &= \left| \sum_{x \in X_i} \sum_{y \in X} \rho_t(y) Q_1(x, y) \pi_i(x) \right| \leq |\rho_t|_\infty \max_{x \in X} \sum_{y \in X} |Q_1(x, y)| \leq Q_\infty |\rho_0|_\infty. \end{aligned}$$

Therefore (2.7) implies

$$|\tilde{\rho}_t(i) - \tilde{\rho}_s(i)| \leq |t - s| Q_\infty |\rho_0|_\infty, \quad 1 \leq i \leq m. \quad (2.17)$$

For equation (2.5) which is written in matrix form, using variation of constants formula, we can obtain

$$\rho_t = e^{(t-s)Q_0/\epsilon} \rho_s + \int_s^t e^{(t-r)Q_0/\epsilon} Q_1 \rho_r dr, \quad 0 \leq s \leq t. \quad (2.18)$$

Since $e^{(t-r)Q_0/\epsilon}$ is a stochastic matrix, we have

$$\begin{aligned} & \left| \int_s^t e^{(t-r)Q_0/\epsilon} Q_1 \rho_r dr \right|_\infty \\ & \leq \int_s^t |Q_1 \rho_r|_\infty dr \leq \int_s^t \|Q_1\|_\infty |\rho_r|_\infty dr \leq (t-s) Q_\infty |\rho_0|_\infty. \end{aligned} \quad (2.19)$$

For the first term on the right hand side of (2.18), noticing Q_0 (therefore also $e^{(t-s)Q_0/\epsilon}$) is a block diagonal matrix and applying Poincaré inequality [2, 3, 5], we deduce

$$\begin{aligned} & |e^{(t-s)Q_0/\epsilon} \rho_s(i, \cdot) - \tilde{\rho}_s(i) \mathbf{1}|_{2, \pi_i} \\ & = |e^{(t-s)Q_0/\epsilon} (\rho_s(i, \cdot) - \tilde{\rho}_s(i) \mathbf{1})|_{2, \pi_i} \leq e^{-\frac{(t-s)\lambda_i}{\epsilon}} |\rho_s(i, \cdot) - \tilde{\rho}_s(i) \mathbf{1}|_{2, \pi_i}, \end{aligned} \quad (2.20)$$

where $1 \leq i \leq m$, $\mathbf{1}$ denotes the constant vector over subset X_i and $\rho_s(i, \cdot)$ denotes the vector consisting of $\rho_s(x)$ for $x \in X_i$. Combining estimates (2.17)-(2.20) together, we have

$$\begin{aligned} & |\rho_t(i, \cdot) - \tilde{\rho}_t(i) \mathbf{1}|_{2, \pi_i} \\ & \leq |\tilde{\rho}_t(i) - \tilde{\rho}_s(i)| + e^{-\frac{(t-s)\lambda_i}{\epsilon}} |\rho_s(i, \cdot) - \tilde{\rho}_s(i) \mathbf{1}|_{2, \pi_i} + (t-s) Q_\infty |\rho_0|_\infty \\ & \leq e^{-\frac{(t-s)\lambda_i}{\epsilon}} |\rho_s(i, \cdot) - \tilde{\rho}_s(i) \mathbf{1}|_{2, \pi_i} + 2(t-s) Q_\infty |\rho_0|_\infty. \end{aligned}$$

Fix index i and define $G(t) = |\rho_t(i, \cdot) - \tilde{\rho}_t(i) \mathbf{1}|_{2, \pi_i}$. We subtract $G(s)$ and then divide $(t-s)$ on both sides of the inequality above. Let $t \rightarrow s+$, we obtain

$$\frac{d^+ G(t)}{dt} + \frac{\lambda_i}{\epsilon} G(t) \leq 2Q_\infty |\rho_0|_\infty, \quad t \geq 0. \quad (2.21)$$

Gronwall's inequality then implies

$$\begin{aligned} |\rho_t(i, \cdot) - \tilde{\rho}_t(i) \mathbf{1}|_{2, \pi_i} & \leq e^{-\frac{\lambda_i t}{\epsilon}} |\rho_0(i, \cdot) - \tilde{\rho}_0(i) \mathbf{1}|_{2, \pi_i} + \frac{2\epsilon}{\lambda_i} Q_\infty |\rho_0|_\infty \\ & \leq \left(e^{-\frac{\lambda_i t}{\epsilon}} + \frac{2\epsilon}{\lambda_i} Q_\infty \right) |\rho_0|_\infty, \end{aligned} \quad (2.22)$$

where we have used

$$\begin{aligned} & |\rho_0(i, \cdot) - \tilde{\rho}_0(i) \mathbf{1}|_{2, \pi_i} \\ & = \left[\sum_{x \in X_i} \left(\rho_0(x) - \sum_{x' \in X_i} \rho_0(x') \pi_i(x') \right)^2 \pi_i(x) \right]^{\frac{1}{2}} \\ & = \left[\sum_{x \in X_i} \rho_0(x)^2 \pi_i(x) - \left(\sum_{x \in X_i} \rho_0(x) \pi_i(x) \right)^2 \right]^{\frac{1}{2}} \leq |\rho_0|_\infty. \end{aligned}$$

2. Now we turn to the second inequality (2.10). First notice that the equation of $\tilde{\rho}_t$ in (2.7)

can be rewritten as

$$\begin{aligned}
\frac{d}{dt}\tilde{\rho}_t(i) &= \sum_{j \neq i} \sum_{x \in X_i, y \in X_j} (\rho_t(y) - \rho_t(x)) Q_1(x, y) \pi_i(x) \\
&= \sum_{j \neq i} (\tilde{\rho}_t(j) - \tilde{\rho}_t(i)) \bar{Q}(i, j) \\
&\quad + \sum_{j \neq i} \sum_{x \in X_i, y \in X_j} \left[(\rho_t(y) - \tilde{\rho}_t(j)) - (\rho_t(x) - \tilde{\rho}_t(i)) \right] Q_1(x, y) \pi_i(x) \\
&= \sum_{j \neq i} (\tilde{\rho}_t(j) - \tilde{\rho}_t(i)) \bar{Q}(i, j) + \phi_t(i) = \bar{Q} \tilde{\rho}_t + \phi_t, \tag{2.23}
\end{aligned}$$

where

$$\begin{aligned}
\phi_t(i) &= \sum_{j \neq i} \sum_{x \in X_i, y \in X_j} \left[(\rho_t(y) - \tilde{\rho}_t(j)) - (\rho_t(x) - \tilde{\rho}_t(i)) \right] Q_1(x, y) \pi_i(x) \\
&= \sum_{j=1}^m \sum_{x \in X_i, y \in X_j} (\rho_t(y) - \tilde{\rho}_t(j)) Q_1(x, y) \pi_i(x),
\end{aligned}$$

since the row sums of Q_1 are zero. Using detailed balance condition (2.1), we can obtain

$$\begin{aligned}
\mathbf{E}_w \phi_t &= \sum_{i=1}^m \phi_t(i) w(i) \\
&= \sum_{i=1}^m \left[\sum_{j=1}^m \sum_{x \in X_i, y \in X_j} (\rho_t(y) - \tilde{\rho}_t(j)) Q_1(x, y) \pi_i(x) \right] w(i) \\
&= \sum_{x \in X} \sum_{y \in X} (\rho_t(y) - \tilde{\rho}_t(j)) \pi(y) Q_1(y, x) = 0.
\end{aligned}$$

We also need to estimate $|\phi_t|_{2,w}$. On one hand, applying inequality (2.22), we can deduce a pointwise estimate

$$|\rho_t(i, x) - \tilde{\rho}_t(i)| \leq \sqrt{\frac{1}{\min_{x' \in X_i} \pi_i(x')}} \left(e^{-\frac{\lambda_i t}{\epsilon}} + \frac{2\epsilon}{\lambda_i} Q_\infty \right) |\rho_0|_\infty, \quad \forall x \in X_i. \tag{2.24}$$

Therefore

$$\begin{aligned}
|\phi_t|_{2,w} &\leq \max_i |\phi_t(i)| \leq Q_\infty \max_i |\rho_t(i, \cdot) - \tilde{\rho}_t(i)|_\infty \\
&\leq Q_\infty \sqrt{\frac{1}{\min_{i, x \in X_i} \pi_i(x)}} \left(e^{-\frac{\lambda_{\min} t}{\epsilon}} + \frac{2\epsilon}{\lambda_{\min}} Q_\infty \right) |\rho_0|_\infty. \tag{2.25}
\end{aligned}$$

On the other hand, we can avoid using pointwise estimate (2.24) and compute

$$\begin{aligned}
|\phi_t|_{2,w}^2 &= \sum_{i=1}^m \left[\sum_{j=1}^m \sum_{x \in X_i, y \in X_j} \left(\rho_t(y) - \tilde{\rho}_t(j) \right) Q_1(x, y) \pi_i(x) \right]^2 w(i) \\
&= \sum_{i=1}^m \left[\sum_{j=1}^m \sum_{x \in X_i, y \in X_j} \left(\rho_t(y) - \tilde{\rho}_t(j) \right) Q_1(y, x) \pi_j(y) w(j) \right]^2 \frac{1}{w(i)} \\
&\leq \left[\sum_{j=1}^m \sum_{y \in X_j} \left(\rho_t(y) - \tilde{\rho}_t(j) \right)^2 \pi_j(y) w(j) \right] \sum_{i=1}^m \left[\sum_{j=1}^m \sum_{y \in X_j} \left(\sum_{x \in X_i} Q_1(y, x) \right)^2 \pi_j(y) w(j) \right] \frac{1}{w(i)} \\
&\leq \left[\sum_{j=1}^m |\rho_t(j, \cdot) - \tilde{\rho}_t(j) \mathbf{1}|_{2, \pi_j}^2 w(j) \right] \sum_{i=1}^m \sum_{j=1}^m \sum_{y \in X_j} \left| \sum_{x \in X_i} Q_1(y, x) \right| \cdot \left| \sum_{x \in X_i} Q_1(x, y) \pi_i(x) \right| \\
&\leq \frac{Q_\infty}{2} \left[\sum_{j=1}^m |\rho_t(j, \cdot) - \tilde{\rho}_t(j) \mathbf{1}|_{2, \pi_j}^2 w(j) \right] \sum_{i=1}^m \sum_{x \in X_i} \sum_{y \in X} |Q_1(x, y)| \pi_i(x) \\
&\leq \frac{m Q_\infty^2}{2} \left[\sum_{j=1}^m |\rho_t(j, \cdot) - \tilde{\rho}_t(j) \mathbf{1}|_{2, \pi_j}^2 w(j) \right],
\end{aligned}$$

where we have used detailed balance condition (2.1) and relation (1.16). Together with (2.25), we could deduce

$$|\phi_t|_{2,w} \leq \left(\min \left\{ \frac{1}{\min_{i, x \in X_i} \pi_i(x)}, \frac{m}{2} \right\} \right)^{\frac{1}{2}} Q_\infty \left(e^{-\frac{\lambda_{\min} t}{\epsilon}} + \frac{2\epsilon}{\lambda_{\min}} Q_\infty \right) |\rho_0|_\infty. \quad (2.26)$$

Now subtract (2.23) by equation (2.8), we obtain

$$\frac{d}{dt} (\tilde{\rho}_t - \bar{\rho}_t) = \bar{Q}(\tilde{\rho}_t - \bar{\rho}_t) + \phi_t, \quad (2.27)$$

together with initial condition $\tilde{\rho}_0 = \bar{\rho}_0$. Therefore we have

$$\tilde{\rho}_t - \bar{\rho}_t = \int_0^t e^{(t-s)\bar{Q}} \phi_s ds. \quad (2.28)$$

Since $\mathbf{E}_w \phi_s = 0$, Poincaré inequality implies

$$|e^{(t-s)\bar{Q}} \phi_s|_{2,w} \leq e^{-\bar{\lambda}(t-s)} |\phi_s|_{2,w}. \quad (2.29)$$

Therefore, using (2.26), we have

$$\begin{aligned}
|\tilde{\rho}_t - \bar{\rho}_t|_{2,w} &\leq \int_0^t |e^{(t-s)\bar{Q}} \phi_s|_{2,w} ds \leq \int_0^t e^{-\bar{\lambda}(t-s)} |\phi_s|_{2,w} ds \\
&\leq \left(\min \left\{ \frac{1}{\min_{i, x \in X_i} \pi_i(x)}, \frac{m}{2} \right\} \right)^{\frac{1}{2}} Q_\infty \int_0^t e^{-\bar{\lambda}(t-s)} |\rho_0|_\infty \left(e^{-\frac{\lambda_{\min} s}{\epsilon}} + \frac{2\epsilon}{\lambda_{\min}} Q_\infty \right) ds \\
&\leq \frac{Q_\infty |\rho_0|_\infty}{\lambda_{\min}} \left(\min \left\{ \frac{1}{\min_{i, x \in X_i} \pi_i(x)}, \frac{m}{2} \right\} \right)^{\frac{1}{2}} \left(\frac{2Q_\infty}{\bar{\lambda}} + 1 \right) \epsilon. \quad \square
\end{aligned}$$

2.3. Poincaré constant, (modified) logarithmic Sobolev constants. In this subsection we consider the asymptotic behavior of the Poincaré constant λ_ϵ , logarithmic Sobolev constant α_ϵ , and modified logarithmic Sobolev constant γ_ϵ defined in (1.5), (1.6) and (1.10),

respectively. We will use the fact that the infima in the definitions can be achieved by some extreme functions. Also notice that in the reversible case, as a generalization of (1.9), we have

$$\begin{aligned}\mathcal{E}_\epsilon(f, g) = & \frac{1}{2\epsilon} \sum_{i=1}^m \sum_{x, x' \in X_i} (f(x') - f(x))(g(x') - g(x))Q_{0,i}(x, x')\pi(x) \\ & + \frac{1}{2} \sum_{i \neq j} \sum_{x \in X_i, y \in X_j} (f(y) - f(x))(g(y) - g(x))Q_1(x, y)\pi(x),\end{aligned}\quad (2.30)$$

for all $f, g : X \rightarrow \mathbb{R}$. See [2, 5] and Appendix A for more details.

We start with the Poincaré constant.

THEOREM 2.4. *Assume Markov chain \mathcal{C} is reversible and $\epsilon \leq 1$. Let $\lambda_\epsilon, \bar{\lambda}$ be the Poincaré constants of Markov chain \mathcal{C} and $\bar{\mathcal{C}}$ corresponding to infinitesimal generator Q and \bar{Q} , respectively. We have*

$$\frac{\bar{\lambda}}{(1 + \epsilon^{\frac{1}{2}})^2} \left(1 - \frac{2\bar{\lambda}\epsilon^{\frac{1}{2}}}{\lambda_{\min}}\right) - \frac{\bar{\lambda}Q_\infty}{\lambda_{\min}}\epsilon^{\frac{1}{2}} \leq \lambda_\epsilon \leq \bar{\lambda}.$$

Proof. Recall the Poincaré constant defined in (1.5)

$$\lambda_\epsilon = \inf_f \left\{ \frac{\mathcal{E}_\epsilon(f, f)}{\text{Var}_\pi f} \mid \text{Var}_\pi f > 0, f : X \rightarrow \mathbb{R} \right\}, \quad (2.31)$$

and the Dirichlet form \mathcal{E}_ϵ in (2.30).

1. First choose function $f : X \rightarrow \mathbb{R}$, s.t. $f(x) = g(i)$ for $x \in X_i$, where g is a function on \bar{X} . Using the fact $\pi(x) = \pi_i(x)w(i)$ when $x \in X_i$, from (2.30) we know

$$\mathcal{E}_\epsilon(f, f) = \frac{1}{2} \sum_{1 \leq i, j \leq m} (g(j) - g(i))^2 \bar{Q}(i, j)w(i) = \bar{\mathcal{E}}(g, g).$$

It is also straightforward to check $\mathbf{E}_\pi f = \mathbf{E}_w g$ and $\text{Var}_\pi f = \text{Var}_w g$. Allowing g to vary among all functions from \bar{X} to \mathbb{R} , we obtain $\lambda_\epsilon \leq \bar{\lambda}$, i.e. the upper bound of the theorem.

2. For the lower bound, we assume the minimum in (2.31) is obtained by function f , i.e.

$$\frac{\mathcal{E}_\epsilon(f, f)}{\text{Var}_\pi f} = \lambda_\epsilon \leq \bar{\lambda}.$$

The estimation of λ_ϵ can be obtained if we could estimate $\mathcal{E}_\epsilon(f, f)$ and the variance of f .

From (2.30), we easily obtain

$$\frac{1}{2\epsilon} \sum_{i=1}^m \left[\sum_{x, x' \in X_i} (f(x') - f(x))^2 Q_{0,i}(x, x')\pi_i(x) \right] w(i) \leq \bar{\lambda} \text{Var}_\pi f. \quad (2.32)$$

Applying Poincaré inequality to Markov chain \mathcal{C}_i for each fixed i , we obtain

$$\sum_{i=1}^m \left[\sum_{x \in X_i} (f(x) - \tilde{f}(i))^2 \pi_i(x) \right] w(i) \leq \frac{\bar{\lambda}}{\lambda_{\min}} \text{Var}_\pi f \epsilon, \quad (2.33)$$

where $\tilde{f}(i) = \sum_{x \in X_i} f(x)\pi_i(x)$. Using the elementary inequality

$$(1 + \epsilon^{\frac{1}{2}})c^2 \geq (a + b + c)^2 - 2(1 + \epsilon^{-\frac{1}{2}})(a^2 + b^2), \quad \forall a, b, c \in \mathbb{R}, \quad (2.34)$$

and the detailed balance condition (2.1), we can estimate the Dirichlet form \mathcal{E}_ϵ in (2.30)

$$\begin{aligned}
& \mathcal{E}_\epsilon(f, f) \\
& \geq \frac{1}{2} \sum_{i \neq j} \sum_{x \in X_i, y \in X_j} (f(y) - f(x))^2 Q_1(x, y) \pi(x) \\
& \geq \frac{1}{1 + \epsilon^{\frac{1}{2}}} \frac{1}{2} \sum_{i \neq j} (\tilde{f}(j) - \tilde{f}(i))^2 \bar{Q}(i, j) w(i) \\
& \quad - \epsilon^{-\frac{1}{2}} \sum_{i \neq j} \sum_{x \in X_i, y \in X_j} \left[(f(x) - \tilde{f}(i))^2 + (\tilde{f}(j) - f(y))^2 \right] Q_1(x, y) \pi(x) \\
& = \frac{1}{1 + \epsilon^{\frac{1}{2}}} \frac{1}{2} \sum_{i \neq j} (\tilde{f}(j) - \tilde{f}(i))^2 \bar{Q}(i, j) w(i) - 2\epsilon^{-\frac{1}{2}} \sum_{i \neq j} \sum_{x \in X_i, y \in X_j} (f(x) - \tilde{f}(i))^2 Q_1(x, y) \pi(x).
\end{aligned} \tag{2.35}$$

Recalling the definition of Q_∞ in (1.16) and applying (2.33), we can estimate the second term on the right hand side of (2.35) and obtain

$$\begin{aligned}
& \sum_{i \neq j} \sum_{x \in X_i, y \in X_j} (f(x) - \tilde{f}(i))^2 Q_1(x, y) \pi(x) \\
& \leq \frac{Q_\infty}{2} \sum_{i=1}^m \sum_{x \in X_i} (f(x) - \tilde{f}(i))^2 \pi(x) \leq \frac{\bar{\lambda} Q_\infty}{2\lambda_{\min}} \text{Var}_\pi f \epsilon.
\end{aligned} \tag{2.36}$$

To estimate the variance of f , we apply inequality (2.33), the elementary inequality

$$(a + b)^2 \leq (1 + \epsilon^{-\frac{1}{2}})a^2 + (1 + \epsilon^{\frac{1}{2}})b^2, \quad \forall a, b \in \mathbb{R},$$

together with the Poincaré inequality for the reduced Markov chain $\bar{\mathcal{C}}$. It gives

$$\begin{aligned}
\text{Var}_\pi f &= \sum_{x \in X} \left(f(x) - \sum_{x' \in X} f(x') \pi(x') \right)^2 \pi(x) \\
&= \sum_{x \in X} \left(f(x) - \sum_{i=1}^m \tilde{f}(i) w(i) \right)^2 \pi(x) \\
&\leq (1 + \epsilon^{-\frac{1}{2}}) \sum_{i=1}^m \sum_{x \in X_i} \left(f(x) - \tilde{f}(i) \right)^2 \pi(x) + (1 + \epsilon^{\frac{1}{2}}) \sum_{i=1}^m \left(\tilde{f}(i) - \sum_{j=1}^m \tilde{f}(j) w(j) \right)^2 w(i) \\
&\leq (1 + \epsilon^{-\frac{1}{2}}) \frac{\bar{\lambda} \epsilon}{\lambda_{\min}} \text{Var}_\pi f + \frac{1}{2} (1 + \epsilon^{\frac{1}{2}}) \bar{\lambda}^{-1} \sum_{1 \leq i, j \leq m} \left(\tilde{f}(j) - \tilde{f}(i) \right)^2 \bar{Q}(i, j) w(i).
\end{aligned} \tag{2.37}$$

Combining (2.35)–(2.37), we arrive at

$$\mathcal{E}_\epsilon(f, f) \geq \frac{\bar{\lambda}}{(1 + \epsilon^{\frac{1}{2}})^2} \text{Var}_\pi f \left[1 - (1 + \epsilon^{-\frac{1}{2}}) \frac{\bar{\lambda} \epsilon}{\lambda_{\min}} \right] - \epsilon^{\frac{1}{2}} \frac{\bar{\lambda} Q_\infty}{\lambda_{\min}} \text{Var}_\pi f,$$

which implies

$$\lambda_\epsilon \geq \frac{\bar{\lambda}}{(1 + \epsilon^{\frac{1}{2}})^2} \left(1 - \frac{2\bar{\lambda} \epsilon^{\frac{1}{2}}}{\lambda_{\min}} \right) - \frac{\bar{\lambda} Q_\infty}{\lambda_{\min}} \epsilon^{\frac{1}{2}}$$

when $\epsilon \leq 1$.

□

We continue to study the logarithmic Sobolev constant.

THEOREM 2.5. *Assume Markov chain \mathcal{C} is reversible. Let $\alpha_\epsilon, \bar{\alpha}$ be the logarithmic Sobolev constants of Markov chain \mathcal{C} and $\bar{\mathcal{C}}$ corresponding to infinitesimal generator Q and \bar{Q} , respectively. We have*

$$\frac{\bar{\alpha}}{1 + \epsilon^{\frac{1}{2}}} \left(1 - \frac{(\bar{\alpha} + \frac{1}{4}Q_\infty)\epsilon}{\alpha_{\min}} \right) - \frac{\bar{\alpha}Q_\infty\epsilon^{\frac{1}{2}}}{2\alpha_{\min}} \leq \alpha_\epsilon \leq \bar{\alpha}.$$

Proof. Recall the logarithmic Sobolev constant defined in (1.6)

$$\alpha_\epsilon = \inf_f \left\{ \frac{\mathcal{E}_\epsilon(f, f)}{\text{Ent}_\pi(f^2)} \mid \text{Ent}_\pi(f^2) > 0, f : X \rightarrow \mathbb{R} \right\}. \quad (2.38)$$

1. The upper bound follows directly by considering functions $f(x) = g(i)$ when $x \in X_i$, $g : \bar{X} \rightarrow \mathbb{R}$, and noticing that $\text{Ent}_\pi(f^2) = \text{Ent}_w(g^2)$, $\mathcal{E}_\epsilon(f, f) = \bar{\mathcal{E}}(g, g)$.
2. For the lower bound, we assume the minimum in (2.38) is achieved with function f , i.e.

$$\alpha_\epsilon = \frac{\mathcal{E}_\epsilon(f, f)}{\text{Ent}_\pi(f^2)} \leq \bar{\alpha}.$$

Estimation of α_ϵ can be obtained if we could estimate both the numerator and denominator. For the Dirichlet form $\mathcal{E}_\epsilon(f, f)$, from (2.30) we have

$$\frac{1}{2\epsilon} \sum_{i=1}^m \sum_{x, x' \in X_i} (f(x') - f(x))^2 Q_{0,i}(x, x') \pi(x) \leq \bar{\alpha} \text{Ent}_\pi(f^2).$$

Applying Poincaré inequality and logarithmic Sobolev inequality for Markov chain \mathcal{C}_i for each fixed i , and noticing $2\alpha_i \leq \lambda_i$ (see (1.11)), we obtain

$$\begin{aligned} \sum_{i=1}^m \left[\sum_{x \in X_i} (f(x) - \tilde{f}(i))^2 \pi_i(x) \right] w(i) &\leq \frac{\bar{\alpha}}{2\alpha_{\min}} \text{Ent}_\pi(f^2) \epsilon, \\ \sum_{i=1}^m \left[\sum_{x \in X_i} |f(x)|^2 \ln \frac{|f(x)|^2}{F(i)^2} \pi_i(x) \right] w(i) &\leq \frac{\bar{\alpha}}{\alpha_{\min}} \text{Ent}_\pi(f^2) \epsilon, \end{aligned} \quad (2.39)$$

respectively. In the above, $F(i)^2 = |f(i, \cdot)|_{2, \pi_i}^2 = \sum_{x \in X_i} |f(x)|^2 \pi_i(x)$ and we have $|\tilde{f}(i)| \leq F(i)$, $1 \leq i \leq m$. We proceed similarly as in the proof of Theorem 2.4. Applying the detailed balance condition (2.1), inequalities (2.34) and (2.39) to the Dirichlet form (2.30), we can obtain

$$\mathcal{E}_\epsilon(f, f) \quad (2.40)$$

$$\begin{aligned} &\geq \frac{1}{2} \sum_{1 \leq i \neq j \leq m} \sum_{x \in X_i, y \in X_j} (f(y) - f(x))^2 Q_1(x, y) \pi(x) \\ &\geq \frac{1}{1 + \epsilon^{\frac{1}{2}}} \frac{1}{2} \sum_{1 \leq i \neq j \leq m} (\tilde{f}(j) - \tilde{f}(i))^2 \bar{Q}(i, j) w(i) - 2\epsilon^{-\frac{1}{2}} \sum_{i=1}^m \sum_{x \in X_i, y \notin X_i} (f(x) - \tilde{f}(i))^2 Q_1(x, y) \pi(x) \\ &\geq \frac{1}{1 + \epsilon^{\frac{1}{2}}} \frac{1}{2} \sum_{1 \leq i \neq j \leq m} (\tilde{f}(j) - \tilde{f}(i))^2 \bar{Q}(i, j) w(i) - Q_\infty \epsilon^{-\frac{1}{2}} \sum_{i=1}^m \sum_{x \in X_i} (f(x) - \tilde{f}(i))^2 \pi(x) \\ &\geq \frac{1}{1 + \epsilon^{\frac{1}{2}}} \frac{1}{2} \sum_{1 \leq i \neq j \leq m} (\tilde{f}(j) - \tilde{f}(i))^2 \bar{Q}(i, j) w(i) - \frac{\bar{\alpha} Q_\infty \text{Ent}_\pi(f^2) \epsilon^{\frac{1}{2}}}{2\alpha_{\min}}. \end{aligned} \quad (2.41)$$

Now we estimate $\text{Ent}_\pi(f^2)$. Using (2.39), the logarithmic Sobolev inequality for the reduced Markov chain $\bar{\mathcal{C}}$, and noticing $|f|_{2,\pi}^2 = |F|_{2,w}^2$, we have

$$\begin{aligned} \text{Ent}_\pi(f^2) &= \sum_{i=1}^m \sum_{x \in X_i} |f(x)|^2 \ln \frac{|f(x)|^2}{|F(i)|^2} \pi_i(x) w(i) + \sum_{i=1}^m |F(i)|^2 \ln \frac{|F(i)|^2}{|f|_{2,\pi}^2} w(i) \\ &\leq \frac{\bar{\alpha}}{\alpha_{\min}} \text{Ent}_\pi(f^2) \epsilon + \frac{1}{2\bar{\alpha}} \sum_{1 \leq i, j \leq m} |F(j) - F(i)|^2 \bar{Q}(i, j) w(i). \end{aligned} \quad (2.42)$$

The first inequality in (2.39) and the definition in (1.3) imply

$$\begin{aligned} 0 &\leq \sum_{i=1}^m \left(F(i)^2 - \tilde{f}(i)^2 \right) w(i) \leq \frac{\bar{\alpha}}{2\alpha_{\min}} \text{Ent}_\pi(f^2) \epsilon, \\ 0 &< \sum_{j \neq i} \bar{Q}(i, j) = \sum_{x \in X_i} \sum_{y \notin X_i} Q_1(x, y) \pi_i(x) \leq \frac{Q_\infty}{2}. \end{aligned}$$

Then the second term on the right hand side of (2.42) can be bounded as

$$\begin{aligned} &\sum_{1 \leq i, j \leq m} \left| F(j) - F(i) \right|^2 \bar{Q}(i, j) w(i) \\ &= \sum_{1 \leq i \neq j \leq m} \left(F(i)^2 - 2F(i)F(j) + F(j)^2 \right) \bar{Q}(i, j) w(i) \\ &\leq 2 \sum_{1 \leq i \neq j \leq m} F(i)^2 \bar{Q}(i, j) w(i) - 2 \sum_{1 \leq i \neq j \leq m} \tilde{f}(i) \tilde{f}(j) \bar{Q}(i, j) w(i) \\ &= \sum_{1 \leq i, j \leq m} \left(\tilde{f}(j) - \tilde{f}(i) \right)^2 \bar{Q}(i, j) w(i) + 2 \sum_{1 \leq i \neq j \leq m} \left(F(i)^2 - \tilde{f}(i)^2 \right) \bar{Q}(i, j) w(i) \\ &\leq \sum_{1 \leq i, j \leq m} \left(\tilde{f}(j) - \tilde{f}(i) \right)^2 \bar{Q}(i, j) w(i) + \frac{Q_\infty \bar{\alpha}}{2\alpha_{\min}} \text{Ent}_\pi(f^2) \epsilon, \end{aligned}$$

where the detailed balance condition (2.4) for Markov chain $\bar{\mathcal{C}}$ has been used.

Therefore (2.42) implies

$$\text{Ent}_\pi(f^2) \leq \frac{\bar{\alpha}}{\alpha_{\min}} \text{Ent}_\pi(f^2) \epsilon + \frac{1}{2\bar{\alpha}} \left[\sum_{1 \leq i, j \leq m} \left(\tilde{f}(j) - \tilde{f}(i) \right)^2 \bar{Q}(i, j) w(i) + \frac{Q_\infty \bar{\alpha}}{2\alpha_{\min}} \text{Ent}_\pi(f^2) \epsilon \right],$$

or equivalently

$$\frac{1}{2} \sum_{1 \leq i, j \leq m} \left(\tilde{f}(j) - \tilde{f}(i) \right)^2 \bar{Q}(i, j) w(i) \geq \bar{\alpha} \text{Ent}_\pi(f^2) \left(1 - \frac{(\bar{\alpha} + \frac{1}{4} Q_\infty) \epsilon}{\alpha_{\min}} \right).$$

Substituting the above inequality into (2.41), we obtain

$$\mathcal{E}_\epsilon(f, f) \geq \frac{\bar{\alpha} \text{Ent}_\pi(f^2)}{1 + \epsilon^{\frac{1}{2}}} \left(1 - \frac{(\bar{\alpha} + \frac{1}{4} Q_\infty) \epsilon}{\alpha_{\min}} \right) - \frac{\bar{\alpha} Q_\infty \text{Ent}_\pi(f^2) \epsilon^{\frac{1}{2}}}{2\alpha_{\min}},$$

which indicates the lower bound.

□

Finally, we study the modified logarithmic Sobolev constant.

THEOREM 2.6. *Let $\gamma_\epsilon, \bar{\gamma}$ be the modified logarithmic Sobolev constants of the reversible Markov chain \mathcal{C} and $\bar{\mathcal{C}}$ corresponding to infinitesimal generator Q and \bar{Q} , respectively. We have*

$$\lim_{\epsilon \rightarrow 0} \gamma_\epsilon = \bar{\gamma}. \quad (2.43)$$

Proof. We argue by contradiction. Suppose the conclusion is not true. First recall the modified logarithmic Sobolev constant defined in (1.10)

$$\gamma_\epsilon = \inf_f \left\{ \frac{\mathcal{E}_\epsilon(f, \ln f)}{\text{Ent}_\pi(f)} \mid \text{Ent}_\pi(f) > 0, f : X \rightarrow \mathbb{R}^+ \right\}. \quad (2.44)$$

Take functions $f(x) = g(i)$ for $x \in X_i$, then it is straightforward to check $\mathcal{E}_\epsilon(f, \ln f) = \bar{\mathcal{E}}(g, \ln g)$ and $\text{Ent}_\pi(f) = \text{Ent}_w(g)$, therefore we can deduce $\gamma_\epsilon \leq \bar{\gamma}$ by allowing function g to vary among all functions $g : \bar{X} \rightarrow \mathbb{R}^+$.

Since $\gamma_\epsilon \leq \bar{\gamma}$, we can find a sequence $\epsilon^{(k)}$, $\lim_{k \rightarrow +\infty} \epsilon^{(k)} = 0$, s.t. $\lim_{k \rightarrow +\infty} \gamma^{(k)} < \bar{\gamma}$. Notice that in this proof, we will use notations $\gamma^{(k)}, \mathcal{E}^{(k)}$ instead of $\gamma_{\epsilon^{(k)}}$ and $\mathcal{E}_{\epsilon^{(k)}}$. We assume the infima in (2.44) are achieved with functions $f_k : X \rightarrow \mathbb{R}^+$, i.e.

$$\gamma^{(k)} = \frac{\mathcal{E}^{(k)}(f_k, \ln f_k)}{\text{Ent}_\pi(f_k)}, \quad \mathbf{E}_\pi f_k = \sum_{x \in X} f_k(x) \pi(x) = 1. \quad (2.45)$$

Let $\pi_{\min} = \min_{x \in X} \pi(x)$. Clearly we have $0 < f_k(x) \leq \pi_{\min}^{-1}$, $\forall x \in X$ (see [2] for the positivity), and therefore

$$0 \leq \text{Ent}_\pi(f_k) \leq \ln \frac{1}{\pi_{\min}}. \quad (2.46)$$

Since f_k are bounded, we further assume they converge to some function $\bar{f} : X \rightarrow \mathbb{R}$ for each $x \in X$ (This can be achieved by considering a convergent subsequence).

From Dirichlet form (2.30) and (2.45), we have

$$\begin{aligned} & \frac{1}{\epsilon^{(k)}} \sum_{i=1}^m \mathcal{E}_i(f_k(i, \cdot), \ln f_k(i, \cdot)) w(i) \\ &= \frac{1}{2\epsilon^{(k)}} \sum_{i=1}^m \sum_{x, x' \in X_i} (f_k(x') - f_k(x)) (\ln f_k(x') - \ln f_k(x)) Q_{0,i}(x, x') \pi(x) \\ &\leq \mathcal{E}^{(k)}(f_k, \ln f_k) = \gamma^{(k)} \text{Ent}_\pi(f_k) \leq \bar{\gamma} \ln \frac{1}{\pi_{\min}}. \end{aligned} \quad (2.47)$$

Recall $\gamma_i > 0$ is the modified logarithmic Sobolev constant of Markov chain \mathcal{C}_i , together with (2.47), we can obtain

$$\sum_{i=1}^m \text{Ent}_{\pi_i}(f_k(i, \cdot)) w(i) \leq \frac{\bar{\gamma} \ln \frac{1}{\pi_{\min}} \epsilon^{(k)}}{\gamma_{\min}}. \quad (2.48)$$

Let $\mu_i^{(k)}$ be the probability measure on X_i s.t. $\mu_i^{(k)}(x) = \frac{f_k(x)}{f_k(i)} \pi_i(x)$, $\forall x \in X_i$. Applying Csiszár-

Kullback-Pinsker inequality, we can obtain

$$\begin{aligned}
& \sum_{i=1}^m \left[\sum_{x \in X_i} |f_k(x) - \tilde{f}_k(i)| \pi_i(x) \right] w(i) = \sum_{i=1}^m \tilde{f}_k(i) \|\mu_i^{(k)} - \pi_i\|_{\text{TV}} w(i) \\
& \leq \sum_{i=1}^m \tilde{f}_k(i) \sqrt{2 \text{Ent}_{\pi_i} \left(\frac{f_k(i, \cdot)}{\tilde{f}_k(i)} \right)} w(i) = \sum_{i=1}^m \sqrt{2 \tilde{f}_k(i) \text{Ent}_{\pi_i}(f_k(i, \cdot))} w(i) \\
& \leq \sqrt{2} \left[\sum_{i=1}^m \text{Ent}_{\pi_i}(f_k(i, \cdot)) w(i) \right]^{\frac{1}{2}} \leq \left(\frac{2\bar{\gamma} \ln \frac{1}{\pi_{\min}} \epsilon^{(k)}}{\gamma_{\min}} \right)^{\frac{1}{2}}, \tag{2.49}
\end{aligned}$$

where $\|\cdot\|_{\text{TV}}$ denotes the total variation distance of two probability measures, and we have used the fact

$$\sum_{i=1}^m \tilde{f}_k(i) w(i) = \sum_{x \in X} f_k(x) \pi(x) = 1. \tag{2.50}$$

Taking the limit $k \rightarrow +\infty$, inequality (2.49) indicates that \bar{f} is constant on each subset X_i , i.e. $\bar{f}(x) = \bar{g}(i)$ if $x \in X_i$, where $\bar{g} : \bar{X} \rightarrow \mathbb{R}^+$. We argue that \bar{f} is both positive and non-constant (this argument is adapted from [2]). Suppose \bar{f} is constant. Notice that

$$\mathbf{E}_{\pi} \bar{f} = \mathbf{E}_w \bar{g} = \lim_{k \rightarrow +\infty} \mathbf{E}_{\pi} f_k = 1, \tag{2.51}$$

therefore we must have $\bar{f} \equiv 1$. Let $f_k = 1 + f'_k$, where $\mathbf{E}_{\pi}(f'_k) = 0$ and $\lim_{k \rightarrow +\infty} f'_k(x) = 0, \forall x \in X$. Using Taylor expansion, we can verify

$$\begin{aligned}
\mathcal{E}^{(k)}(f_k, \ln f_k) &= \left(1 + \mathcal{O}(|f'_k|_{\infty})\right) \mathcal{E}^{(k)}(f'_k, f'_k), \\
\text{Ent}_{\pi} f_k &= \frac{1}{2} \text{Var}_{\pi} f'_k + \mathcal{O}(|f'_k|_{\infty}^3).
\end{aligned}$$

Because $\text{Var}_{\pi} f'_k \geq |f'_k|_{\infty}^2 \pi_{\min}$, we know $\text{Ent}_{\pi} f_k = \frac{1+o(1)}{2} \text{Var}_{\pi} f'_k$. Applying Poincaré inequality and Theorem 2.4 (which implies $\lambda^{(k)}$ converges to $\bar{\lambda}$), we can estimate

$$\bar{\gamma} > \lim_{k \rightarrow +\infty} \gamma^{(k)} \geq \liminf_{k \rightarrow \infty} \frac{\mathcal{E}^{(k)}(f'_k, f'_k)}{\frac{1}{2} \text{Var}_{\pi}(f'_k)} \geq 2 \lim_{k \rightarrow +\infty} \lambda^{(k)} = 2\bar{\lambda}, \tag{2.52}$$

which contradicts to the relation (1.13). Therefore function \bar{f} is non-constant. Now we consider the positiveness. Define two disjoint sets

$$M = \{x \in X \mid \bar{f}(x) = 0\}, \quad M' = \{x \in X \mid \bar{f}(x) > 0\}. \tag{2.53}$$

Since $\mathbf{E}_{\pi} \bar{f} = 1$, we know M' is not empty. Now assume set M is also nonempty. Applying the irreducibility of Markov chain \mathcal{C} to subset M and M' , we conclude that $\exists x \in M, y \in M'$ s.t. $Q(x, y) > 0$. Since \bar{f} is constant on each subset X_i , we know $x \in X_i, y \in X_j$, for some $i \neq j$ and $Q_1(x, y) > 0$. Then we have

$$\begin{aligned}
+\infty &= \lim_{k \rightarrow +\infty} (f_k(y) - f_k(x)) (\ln f_k(y) - \ln f_k(x)) Q_1(x, y) \pi(x) \\
&\leq \limsup_{k \rightarrow +\infty} \mathcal{E}^{(k)}(f_k, \ln f_k) = \limsup_{k \rightarrow +\infty} \gamma^{(k)} \text{Ent}_{\pi}(f_k) \leq \bar{\gamma} \ln \frac{1}{\pi_{\min}}.
\end{aligned}$$

This contradiction shows that M is empty and therefore \bar{f} is positive. Now we can take the limits

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \mathcal{E}^{(k)}(f_k, \ln f_k) &\geq \bar{\mathcal{E}}(\bar{g}, \ln \bar{g}) \\ \lim_{k \rightarrow +\infty} \text{Ent}_\pi(f_k) &= \text{Ent}_\pi(\bar{f}) = \text{Ent}_w(\bar{g}), \end{aligned}$$

and

$$\bar{\gamma} > \lim_{k \rightarrow +\infty} \gamma^{(k)} = \lim_{k \rightarrow +\infty} \frac{\mathcal{E}^{(k)}(f_k, \ln f_k)}{\text{Ent}_\pi(f_k)} \geq \frac{\bar{\mathcal{E}}(\bar{g}, \ln \bar{g})}{\text{Ent}_w(\bar{g})}. \quad (2.54)$$

But the above inequality is in contradiction with the fact that $\bar{\gamma}$ is the modified logarithmic Sobolev constant of the reduced Markov chain \bar{C} . Therefore we conclude $\lim_{\epsilon \rightarrow 0} \gamma_\epsilon = \bar{\gamma}$. \square

REMARK 1. *Theorem 2.4 and Theorem 2.5 imply that both the Poincaré constant λ_ϵ and logarithmic Sobolev constant α_ϵ of Markov chain \mathcal{C} converge to their counterparts $\bar{\lambda}$, $\bar{\alpha}$ of the reduced Markov chain \bar{C} and the convergence order is $\mathcal{O}(\epsilon^{\frac{1}{2}})$.*

On the other hand, the result of Theorem 2.6 is weaker in that we obtain convergence without convergence order.

3. Asymptotic analysis : general case. In this section we consider the general case without assuming reversibility. A convergence result of Kolmogorov backward equation can be found in [21] and will not be discussed here (also see Appendix B). Unlike the reversible case in Section 2, relation (2.2) does not hold in general and the invariant measure π^ϵ will depend on parameter ϵ . Instead, we have the following result (recall the notations in Subsection 1.2).

THEOREM 3.1. *Let π^ϵ , w , π_i be the invariant measures of Markov chain \mathcal{C} , \bar{C} , and \mathcal{C}_i , respectively. We have*

$$\left(\sum_{i=1}^m \sum_{x \in X_i} |\pi^\epsilon(x) - w(i)\pi_i(x)|^2 \right)^{\frac{1}{2}} \leq 3^{\frac{1}{2}} \left[\frac{1}{2} + \left(\frac{1}{2} + m \right) \bar{\sigma}^{-2} d\Gamma + n \right]^{\frac{1}{2}} \sigma_{\min}^{-1} Q_\infty \epsilon.$$

Proof. We first study the invariant measure π^ϵ , which satisfies equation

$$\left(\frac{Q_0^T}{\epsilon} + Q_1^T \right) \pi^\epsilon = 0, \quad (3.1)$$

i.e. $Q_0^T \pi^\epsilon = -\epsilon Q_1^T \pi^\epsilon$. Since matrix Q_0 is a block diagonal matrix given in (1.2), we obtain linear systems

$$Q_{0,i}^T \pi^\epsilon(i, \cdot) = -\epsilon b(i, \cdot), \quad 1 \leq i \leq m, \quad (3.2)$$

where $b(i, x) = \sum_{y \in X} Q_1(y, x) \pi^\epsilon(y)$, $x \in X_i$. Summing up $x \in X_i$ in (3.2) and noticing that each matrix $Q_{0,i}$ have zero row sums, we obtain

$$\sum_{x \in X_i} b(i, x) = \sum_{x \in X_i} \sum_{y \in X} Q_1(y, x) \pi^\epsilon(y) = 0, \quad 1 \leq i \leq m. \quad (3.3)$$

Using (1.16), we can estimate

$$\begin{aligned}
|b(i, \cdot)|_2 &= \left[\sum_{x \in X_i} \left(\sum_{y \in X} Q_1(y, x) \pi^\epsilon(y) \right)^2 \right]^{\frac{1}{2}} \leq \left[\sum_{x \in X_i} \sum_{y \in X} Q_1^2(y, x) \pi^\epsilon(y) \right]^{\frac{1}{2}} \\
&\leq \left[\max_{x, y \in X} |Q_1(x, y)| \sum_{x \in X_i} \sum_{y \in X} |Q_1(y, x)| \pi^\epsilon(y) \right]^{\frac{1}{2}} \\
&\leq \left[\frac{1}{2} Q_\infty \sum_{x \in X_i} \sum_{y \in X} |Q_1(y, x)| \pi^\epsilon(y) \right]^{\frac{1}{2}} =: g(i),
\end{aligned}$$

which implies

$$\sum_{i=1}^m |b(i)|_2^2 \leq \sum_{i=1}^m g(i)^2 \leq \frac{1}{2} Q_\infty^2. \quad (3.4)$$

Applying Lemma A.1 to equation (3.2), we have

$$|\pi^\epsilon(i, \cdot) - w^\epsilon(i) \pi_i|_\infty \leq |\pi^\epsilon(i, \cdot) - w^\epsilon(i) \pi_i|_2 \leq \epsilon \sigma_i^{-1} g(i) \quad (3.5)$$

for some $w^\epsilon(i) \in \mathbb{R}$. Recall that σ_i is the smallest nonzero singular value of matrix $Q_{0,i}$. Let $\pi^\epsilon(i, \cdot) = w^\epsilon(i) \pi_i(\cdot) + r^\epsilon(i, \cdot)$, $1 \leq i \leq m$, using (3.3), we have

$$\begin{aligned}
\sum_{j=1}^m \bar{Q}(j, i) w^\epsilon(j) &= \sum_{x \in X_i} \sum_{j=1}^m \sum_{y \in X_j} Q_1(y, x) \pi_i(y) w^\epsilon(j) \\
&= - \sum_{x \in X_i} \sum_{j=1}^m \sum_{y \in X_j} Q_1(y, x) r^\epsilon(j, y) =: \bar{b}^\epsilon(i).
\end{aligned} \quad (3.6)$$

We also have

$$\begin{aligned}
|\bar{b}^\epsilon|_2 &= \left[\sum_{i=1}^m \left| \sum_{x \in X_i} \sum_{j=1}^m \sum_{y \in X_j} Q_1(y, x) r^\epsilon(j, y) \right|^2 \right]^{\frac{1}{2}} \\
&\leq \left[\sum_{i=1}^m \left(\sum_{x \in X_i} \sum_{y \in X} Q_1(y, x)^2 \right) \left(\sum_{j=1}^m \sum_{y \in X_j} \sum_{x \in X_i, y \sim x} r^\epsilon(j, y)^2 \right) \right]^{\frac{1}{2}},
\end{aligned} \quad (3.7)$$

where $y \sim x$ means $Q_1(y, x) \neq 0$. From (3.4) and (3.5),

$$\left(\sum_{j=1}^m \sum_{y \in X_j} \sum_{x \in X_i, y \sim x} r^\epsilon(j, y)^2 \right)^{\frac{1}{2}} \leq \left(d \sum_{j=1}^m |r^\epsilon(j, \cdot)|_2^2 \right)^{\frac{1}{2}} \leq \left(\frac{d}{2} \right)^{\frac{1}{2}} \sigma_{\min}^{-1} Q_\infty \epsilon,$$

therefore (3.7) becomes

$$|\bar{b}^\epsilon|_2 \leq \epsilon \sigma_{\min}^{-1} Q_\infty \left(\frac{d\Gamma}{2} \right)^{\frac{1}{2}},$$

where $\Gamma = \text{tr}(Q_1 Q_1^T)$. Now applying Lemma A.1 to equation (3.6), we obtain

$$|w^\epsilon - \lambda w|_\infty \leq |w^\epsilon - \lambda w|_2 \leq (\bar{\sigma} \sigma_{\min})^{-1} Q_\infty \left(\frac{d\Gamma}{2} \right)^{\frac{1}{2}} \epsilon, \quad (3.8)$$

where $\lambda \in \mathbb{R}$, w is the invariant measure of the reduced Markov chain $\bar{\mathcal{C}}$, i.e. $\bar{Q}^T w = 0$ and $\bar{\sigma}$ is the smallest nonzero singular value of \bar{Q} . Therefore

$$\left| \sum_{i=1}^m (w^\epsilon(i) - \lambda w(i)) \right| \leq m^{\frac{1}{2}} |w^\epsilon - \lambda w|_2 \leq (\bar{\sigma} \sigma_{\min})^{-1} Q_\infty \left(\frac{m d \Gamma}{2} \right)^{\frac{1}{2}} \epsilon. \quad (3.9)$$

From (3.4), (3.5) and

$$\sum_{i=1}^m w^\epsilon(i) = 1 - \sum_{i=1}^m \sum_{x \in X_i} r^\epsilon(i, x), \quad \sum_{i=1}^m w(i) = 1,$$

we can estimate

$$\begin{aligned} \left| \sum_{i=1}^m (w^\epsilon(i) - \lambda w(i)) \right| &= \left| 1 - \lambda - \sum_{i=1}^m \sum_{x \in X_i} r^\epsilon(i, x) \right| \geq |1 - \lambda| - \left| \sum_{i=1}^m \sum_{x \in X_i} r^\epsilon(i, x) \right|, \\ \left| \sum_{i=1}^m \sum_{x \in X_i} r^\epsilon(i, x) \right| &\leq \sum_{i=1}^m \left| \sum_{x \in X_i} r^\epsilon(i, x) \right| \leq \sum_{i=1}^m n_i^{\frac{1}{2}} \sigma_i^{-1} g(i) \epsilon \leq \left(\frac{n}{2} \right)^{\frac{1}{2}} \sigma_{\min}^{-1} Q_\infty \epsilon. \end{aligned}$$

Together with (3.9), we know

$$|1 - \lambda| \leq \left[\bar{\sigma}^{-1} \left(\frac{m d \Gamma}{2} \right)^{\frac{1}{2}} + \left(\frac{n}{2} \right)^{\frac{1}{2}} \right] \sigma_{\min}^{-1} Q_\infty \epsilon. \quad (3.10)$$

Combining (3.5), (3.8) and (3.10), we have

$$\begin{aligned} &\left(\sum_{i=1}^m \sum_{x \in X_i} |\pi^\epsilon(x) - w(i) \pi_i(x)|^2 \right)^{\frac{1}{2}} \\ &\leq 3^{\frac{1}{2}} \left[\sum_{i=1}^m |\pi^\epsilon(i, \cdot) - w^\epsilon(i) \pi_i|^2_2 + |w^\epsilon - \lambda w|_2^2 + |\lambda - 1|^2 \sum_{i=1}^m w(i)^2 \right]^{\frac{1}{2}} \\ &\leq 3^{\frac{1}{2}} \left\{ \frac{1}{2} + \frac{\bar{\sigma}^{-2} d \Gamma}{2} + \left[\bar{\sigma}^{-1} \left(\frac{m d \Gamma}{2} \right)^{\frac{1}{2}} + \left(\frac{n}{2} \right)^{\frac{1}{2}} \right]^2 \right\}^{\frac{1}{2}} \sigma_{\min}^{-1} Q_\infty \epsilon \\ &\leq 3^{\frac{1}{2}} \left[\frac{1}{2} + \left(\frac{1}{2} + m \right) \bar{\sigma}^{-2} d \Gamma + n \right]^{\frac{1}{2}} \sigma_{\min}^{-1} Q_\infty \epsilon. \end{aligned}$$

□

Based on Theorem 3.1, we can obtain convergence results of various constants of Markov chain \mathcal{C} . In the proof of the following result, we will use the fact that the infima in definitions (1.5), (1.6) and (1.10) can be attained by some functions. This fact can be verified using arguments in [2], which is also valid in non-reversible case.

THEOREM 3.2. *Let λ_ϵ , α_ϵ , γ_ϵ be the Poincaré constant, logarithmic Sobolev constant and modified logarithmic Sobolev constant of Markov chain \mathcal{C} . Also let $\bar{\lambda}$, $\bar{\alpha}$, $\bar{\gamma}$ be their counterparts of Markov chain $\bar{\mathcal{C}}$. We have*

$$\lim_{\epsilon \rightarrow 0} \lambda_\epsilon = \bar{\lambda}, \quad \lim_{\epsilon \rightarrow 0} \alpha_\epsilon = \bar{\alpha}, \quad \lim_{\epsilon \rightarrow 0} \gamma_\epsilon = \bar{\gamma}. \quad (3.11)$$

Proof. We will sketch the proof, since the argument is similar to Theorem 2.6.

1. First consider the Poincaré constant. Let function $g : \bar{X} \rightarrow \mathbb{R}$ satisfy $\mathbf{E}_w g = 0$ and $\text{Var}_w g = 1$. Define $f(x) = g(i)$ for $x \in X_i$. We know $|f|_\infty = |g|_\infty$ is bounded. Applying Theorem 3.1 and using (1.9), we know

$$\mathcal{E}_\epsilon(f, f) = \frac{1}{2} \sum_{i \neq j} \sum_{x \in X_i} \sum_{y \in X_j} (f(y) - f(x))^2 Q_1(x, y) \pi^\epsilon(x) = \bar{\mathcal{E}}(g, g) + \mathcal{O}(\epsilon), \quad (3.12)$$

$$\text{Var}_{\pi^\epsilon} f = \sum_{i=1}^m g^2(i) \left(\sum_{x \in X_i} \pi^\epsilon(x) \right) - \left(\sum_{i=1}^m g(i) \sum_{x \in X_i} \pi^\epsilon(x) \right)^2 = \text{Var}_w g + \mathcal{O}(\epsilon). \quad (3.13)$$

Therefore

$$\limsup_{\epsilon \rightarrow 0} \lambda_\epsilon \leq \limsup_{\epsilon \rightarrow 0} \frac{\mathcal{E}_\epsilon(f, f)}{\text{Var}_{\pi^\epsilon} f} = \frac{\bar{\mathcal{E}}(g, g)}{\text{Var}_w g},$$

and we obtain $\limsup_{\epsilon \rightarrow 0} \lambda_\epsilon \leq \bar{\lambda}$ after taking infimum among functions $g : \bar{X} \rightarrow \mathbb{R}$.

Now suppose the conclusion is not true, then we can find a sequence $\epsilon^{(k)}$, $\lim_{k \rightarrow +\infty} \epsilon^{(k)} = 0$ and $\lim_{k \rightarrow +\infty} \lambda^{(k)} < \bar{\lambda}$ (We use the same notations as in Theorem 2.6). Let function f_k be the extreme functions in (1.5) and satisfy $\text{Var}_{\pi^{(k)}} f_k = 1$ and $\mathbf{E}_{\pi^{(k)}} f_k = 0$. Then $\lambda^{(k)} = \mathcal{E}^{(k)}(f_k, f_k)$. It is easy to see

$$\limsup_{k \rightarrow +\infty} |f_k|_\infty < +\infty,$$

and therefore we can find a subsequence (also denoted as f_k for simplicity) s.t. f_k converges to $f : X \rightarrow \mathbb{R}$. Using $\lim_{k \rightarrow +\infty} \mathcal{E}^{(k)}(f_k, f_k) \leq \bar{\lambda}$, we can deduce that $f(x) = g(i)$ if $x \in X_i$, for some $g : \bar{X} \rightarrow \mathbb{R}$. And $\text{Var}_w g = 1$, $\mathbf{E}_w g = 0$. Therefore

$$\bar{\lambda} \leq \bar{\mathcal{E}}(g, g) \leq \lim_{k \rightarrow +\infty} \mathcal{E}^{(k)}(f_k, f_k) = \lim_{k \rightarrow +\infty} \lambda^{(k)} < \bar{\lambda}. \quad (3.14)$$

This contradiction shows that $\lim_{\epsilon \rightarrow 0} \lambda_\epsilon = \bar{\lambda}$.

2. We continue to prove the convergence of the modified logarithmic Sobolev constant γ_ϵ (the proof for the convergence of α_ϵ is similar and is omitted). First of all, using a similar argument as above, we can obtain

$$\limsup_{\epsilon \rightarrow 0} \gamma_\epsilon \leq \bar{\gamma}. \quad (3.15)$$

Suppose the conclusion is not true and then we can find sequence $\epsilon^{(k)}$, s.t. $\lim_{k \rightarrow +\infty} \epsilon^{(k)} = 0$ and $\lim_{k \rightarrow +\infty} \gamma^{(k)} < \bar{\gamma}$. Let f_k be the extreme functions in (1.10) with $\mathbf{E}_{\pi^{(k)}} f_k = 1$. We can argue as above that $\limsup_{k \rightarrow +\infty} |f_k|_\infty < +\infty$, and there is a subsequence (also denoted as f_k) converging to some function f . Using (1.9) and Lemma 2.7 in [5], we have

$$\begin{aligned} & \frac{1}{2\epsilon^{(k)}} \sum_{i=1}^m \sum_{x, x' \in X_i} (f_k^{\frac{1}{2}}(x') - f_k^{\frac{1}{2}}(x))^2 Q_{0,i}(x, x') \pi^{(k)}(x) \\ & \leq \mathcal{E}^{(k)}(f_k^{\frac{1}{2}}, f_k^{\frac{1}{2}}) \leq \frac{1}{2} \mathcal{E}^{(k)}(f_k, \ln f_k) = \frac{\lambda^{(k)}}{2} \text{Ent}_{\pi^{(k)}}(f_k). \end{aligned} \quad (3.16)$$

Taking limit $k \rightarrow +\infty$, applying Theorem 3.1 and the boundness of $\text{Ent}_{\pi^{(k)}}(f_k)$, we can deduce that f is constant on each subset X_i , i.e. $f(x) = g(i)$ if $x \in X_i$, for some $g : \bar{X} \rightarrow \mathbb{R}^+$. We have $\mathbf{E}_w g = \lim_{k \rightarrow +\infty} \mathbf{E}_{\pi^{(k)}} f_k = 1$. The same argument as in Theorem 2.6 shows that g is positive. Now we show g is non-constant. Assume g is constant and let $f_k = 1 + f'_k$, then we have $\mathbf{E}_{\pi^{(k)}} f'_k = 0$ and $\lim_{k \rightarrow +\infty} f'_k(x) = 0$, $\forall x \in X$. Using Taylor expansion, we have

$$\begin{aligned} \mathcal{E}^{(k)}(f_k, \ln f_k) &= - \sum_{x \in X} f_k(x) \left[\sum_{y \in X, y \neq x} Q(x, y) (\ln f_k(y) - \ln f_k(x)) \right] \pi^{(k)}(x) \\ &= (1 + \mathcal{O}(|f'_k|_\infty)) \mathcal{E}^{(k)}(f'_k, f'_k) \end{aligned}$$

and $\text{Ent}_{\pi^{(k)}}(f_k) = (\frac{1+o(1)}{2})\text{Var}_{\pi^{(k)}} f'_k$. We can deduce a contradiction as in Theorem 2.6. Therefore g is non-constant, i.e. $\text{Ent}_w g > 0$. Taking the limit, we obtain

$$\bar{\gamma} \leq \frac{\bar{\mathcal{E}}(g, \ln g)}{\text{Ent}_w(g)} \leq \lim_{k \rightarrow +\infty} \frac{\mathcal{E}^{(k)}(f_k, \ln f_k)}{\text{Ent}_{\pi^{(k)}}(f_k)} = \lim_{k \rightarrow +\infty} \gamma^{(k)} < \bar{\gamma}. \quad (3.17)$$

This contradiction shows that $\lim_{\epsilon \rightarrow 0} \gamma_\epsilon = \bar{\gamma}$.

□

4. Conclusion. In this paper we consider continuous-time Markov chains on finite state space and focus on the situation when systems' transitions within clusters are much faster than transitions among clusters. Several asymptotic results are obtained concerning Kolmogorov backward equation, Poincaré constant, and (modified) logarithmic Sobolev constants. These results validate the reduced Markov chain as an approximation of the multiscale Markov chain in the asymptotic limit. Especially, when understanding the multiscale Markov chain becomes infeasible, either due to an extremely large state space or limited information to identify all transition rates, our results will be instructive as it suggests that the reduced Markov chain can be a useful approximation of the original one.

On the other hand, while we assume that there are several subsets of the state space such that transitions between them are relatively slow, in applications it might be the case that these subsets are not known a priori and need to be identified. How to identify (clustering) the slow subsets is an important problem in the studies of proteins [4, 22], principal component analysis [11, 1], climates [16] and network [8, 18] et al. Readers are referred to those literatures for more details.

In future work, it might be interesting to consider asymptotic behaviors of other constants in [7, 15]. As more and more real data become available nowadays, it is also interesting to quantify the approximation error of the reduced Markov chain using a data-based approach.

Acknowledgement. Part of this work was done when the author was visiting School of Mathematics Sciences at Peking University and institute of natural sciences at Shanghai Jiao Tong University. The hospitality during the stay is gratefully acknowledged.

Appendix A. Some useful facts. In this section we collect some results related to continuous-time Markov chain. Let $n > 1$, Q be an $n \times n$ matrix satisfying $Q_{ij} \geq 0$ for $1 \leq i \neq j \leq n$ and $Q_{ii} = -\sum_{j \neq i} Q_{ij}$, $1 \leq i \leq n$. We will call such matrix as transition rate matrix. Clearly, $Q\mathbf{1} = 0$, where $\mathbf{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^n$. Define $P(t) = e^{tQ}$, then $P(t)_{ij} \geq 0$ for $1 \leq i, j \leq n$, $P(t)P(s) = P(t+s)$, for $t, s \geq 0$ and $P(t)\mathbf{1} = \mathbf{1}$. Therefore $P(t)$ are stochastic matrices and satisfy semigroup property. Let $\Omega = \{1, 2, \dots, n\}$ and $\mathcal{F} = \{f \mid f : \Omega \rightarrow \mathbb{R}\}$, which can be viewed as \mathbb{R}^n . For $f \in \mathcal{F}$, denote $|f|_\infty = \max_{i \in \Omega} |f(i)|$. Then $P(t)$ defines a semigroup on \mathcal{F} with infinitesimal generator Q . It also defines a continuous-time Markov chain $x(t)$ on Ω such that $\mathbf{P}(x(t) = j \mid x(0) = i) = P(t)_{ij}$, $1 \leq i, j \leq n$. Define $f_t = P(t)f \in \mathcal{F}$ for function $f \in \mathcal{F}$, we have

$$f_t(i) = \mathbf{E}(f(x(t)) \mid x(0) = i) \quad (\text{A.1})$$

and it satisfies the Kolmogorov backward equation

$$\frac{d}{dt} f_t = Q f_t, \quad f_0 = f. \quad (\text{A.2})$$

From (A.1), we know $|f_t|_\infty \leq |f_0|_\infty$.

Assume the probability distribution of $x(t)$ at time $t \geq 0$ is μ_t , then it is known that μ_t satisfies Kolmogorov forward (Fokker-Planck) equation

$$\dot{\mu}_t = Q^T \mu_t \quad (\text{A.3})$$

with initial value μ_0 . Therefore we have $\mu_t = P(t)^T \mu_0$. A probability measure π is called the invariant measure of Markov chain x_t iff $Q^T \pi = 0$. If we further assume the Markov chain is irreducible, then the invariant measure is unique. Also define the π -weighted inner product on \mathcal{F} as

$$\langle f, g \rangle_\pi = \sum_{i \in \Omega} f(i)g(i)\pi(i), \quad f, g \in \mathcal{F},$$

and the Dirichlet form \mathcal{E}

$$\mathcal{E}(f, g) = -\langle f, Qg \rangle_\pi.$$

Let Q^* be the adjoint matrix under $\langle \cdot, \cdot \rangle_\pi$, we have $Q_{ij}^* = \frac{Q_{ji}\pi(j)}{\pi(i)}$. We can check that Q^* is also a transition rate matrix and $(Q^*)^T \pi = 0$. The corresponding Markov chain defined by Q^* is called the time-reversed Markov chain. For $f \in \mathcal{F}$, we have

$$\begin{aligned} \mathcal{E}(f, f) &= -\left\langle \frac{Q + Q^*}{2} f, f \right\rangle_\pi \\ &= \frac{1}{2} \sum_{i, j \in \Omega} \frac{Q_{ij} + Q_{ij}^*}{2} (f(i) - f(j))^2 \pi(i) \\ &= \frac{1}{2} \sum_{i, j \in \Omega} Q_{ij} (f(i) - f(j))^2 \pi(i) \geq 0. \end{aligned} \quad (\text{A.4})$$

Define matrix

$$\Pi = \text{diag}\{\pi(1), \pi(2), \dots, \pi(n)\},$$

then we have the matrix equation $Q^* = \Pi^{-1} Q^T \Pi$. It is direct to see that

$$\begin{aligned} &\text{Dirichlet form } \mathcal{E} \text{ is symmetric} \\ \iff &Q = Q^* \\ \iff &\pi(i)Q_{ij} = \pi(j)Q_{ji}, \forall i, j \in \Omega. \end{aligned}$$

In this case, we say π satisfies the detailed balance condition and the Markov chain is reversible.

For μ_t satisfying (A.3), we define $\mu_t = \rho_t \pi$, i.e. $\mu_t(i) = \rho_t(i)\pi(i)$, $i \in \Omega$, then

$$\frac{d}{dt} \rho_t = Q^* \rho_t. \quad (\text{A.5})$$

When $Q^* = Q$, i.e. the detailed balance condition holds, equation (A.5) coincides with the Kolmogorov backward equation (A.2).

Consider the singular value decomposition (SVD) $Q = UDV^T$, where U, V are $n \times n$ orthogonal matrix. $D = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ is a diagonal matrix consisting of singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$. Denote i th column of matrix U, V as U_i, V_i , respectively, i.e.

$U = [U_1, U_2, \dots, U_n], V = [V_1, V_2, \dots, V_n]$. Then $\{U_i\}_{1 \leq i \leq n}$ and $\{V_i\}_{1 \leq i \leq n}$ are two orthonormal basis of \mathbb{R}^n . Since $Q\mathbf{1} = 0$, $Q^T\pi = 0$ and using the fact that the invariant measure π is unique, we know $\sigma_n = 0 < \sigma_{n-1}$. We can further deduce that $V_n \parallel \mathbf{1}$ and $U_n \parallel \pi$. The linear system $Q^T x = b$ can be studied based on the SVD decomposition. We have

LEMMA A.1. *Consider linear system $Q^T x = b$, where $x, b \in \mathbb{R}^n$.*

1. *There is a solution if and only if $b^T \mathbf{1} = 0$.*
2. *Assume $b^T \mathbf{1} = 0$, then the solutions can be written as*

$$x = a\pi + \sum_{i=1}^{n-1} \sigma_i^{-1} (V_i^T b) U_i, \quad (\text{A.6})$$

for $\forall a \in \mathbb{R}$. Furthermore,

$$|x - a\pi|_\infty \leq |x - a\pi|_2 \leq \sigma_{n-1}^{-1} |b|_2. \quad (\text{A.7})$$

Proof.

1. Assume $Q^T x = b$, we have $b^T \mathbf{1} = x^T Q \mathbf{1} = 0$. The sufficiency follows from the second conclusion.
2. We directly verify that expression (A.6) satisfies the equation $Q^T x = b$. $\forall a \in \mathbb{R}$, using $Q^T \pi = 0$, $Q = UDV^T$ and U_i, V_i are orthonormal, we have

$$\begin{aligned} Q^T x &= Q^T \left(a\pi + \sum_{i=1}^{n-1} \sigma_i^{-1} (V_i^T b) U_i \right) \\ &= \sum_{i=1}^{n-1} \sigma_i^{-1} (V_i^T b) V D U^T U_i \\ &= \sum_{i=1}^{n-1} (V_i^T b) V_i = b. \end{aligned}$$

In the last equality, we have used $b^T V_n = b^T \mathbf{1} = 0$.

On the contrary, suppose $x \in \mathbb{R}^n$ satisfies the equation $Q^T x = b$. Since U_i is orthonormal and $U_n \parallel \pi$, we can assume $x = a\pi + \sum_{i=1}^{n-1} a_i U_i$. Substituting it into $Q^T x = b$, we obtain $a_i = \sigma_i^{-1} (V_i^T b)$, $1 \leq i \leq n-1$, i.e. x is given by (A.6).

And we can estimate

$$|x - a\pi|_2 = \left| \sum_{i=1}^{n-1} \sigma_i^{-1} (V_i^T b) U_i \right|_2 = \left(\sum_{i=1}^{n-1} \sigma_i^{-2} (V_i^T b)^2 \right)^{\frac{1}{2}} \leq \sigma_{n-1}^{-1} \left(\sum_{i=1}^{n-1} (V_i^T b)^2 \right)^{\frac{1}{2}} = \sigma_{n-1}^{-1} |b|_2, \quad (\text{A.8})$$

i.e. (A.7) is proved.

□

In the reversible case, we have $Q = Q^*$ and $\Pi Q = Q^T \Pi$, which indicates that $\Pi^{\frac{1}{2}} Q \Pi^{-\frac{1}{2}}$ is symmetric. Consider the eigenvalue decomposition $\Pi^{\frac{1}{2}} Q \Pi^{-\frac{1}{2}} = U D U^T$ with $U^T U = U U^T = I$, $D = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is a diagonal matrix consisting of real eigenvalues. Then $Q = T D T^{-1}$, with $T = \Pi^{-\frac{1}{2}} U$. Denote i th column vector of T as T_i , i.e. $T = [T_1, T_2, \dots, T_n]$, then we have $Q T_i = \lambda_i T_i$, $\langle T_i, T_j \rangle_\pi = \delta_{ij}$. Therefore T_i is the eigenvector of Q corresponding to eigenvalue

λ_i and $\{T_i\}_{1 \leq i \leq n}$ forms an orthonormal basis of $L^2_\pi(\Omega)$. For two functions f, g on Ω written as $f = \sum_{i=1}^n f_i T_i$, $g = \sum_{i=1}^n g_i T_i$, we have

$$\mathcal{E}(f, g) = -\langle f, Qg \rangle_\pi = -\sum_{i=1}^n \lambda_i f_i g_i.$$

From (A.4), we could assume $0 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n$. It is clear that the Poincaré constant $\lambda = -\lambda_2 > 0$.

Appendix B. Asymptotic expansion method : formal argument. Asymptotic expansion method has been widely used in studying dynamical systems, partial differential equations in certain limiting regime, see [6, 23, 20, 21]. In this section, we consider the Kolmogorov backward equation and various constants studied in Section 2-3 using this method. First consider the equation

$$\frac{d}{dt} \rho_t = Q \rho_t = \left(\frac{1}{\epsilon} Q_0 + Q_1 \right) \rho_t. \quad (\text{B.1})$$

Assume we have the expansion

$$\rho_t = \rho_{t,0} + \epsilon \rho_{t,1} + \epsilon^2 \rho_{t,2} + \dots. \quad (\text{B.2})$$

Substitute it into (B.1), we obtain

$$\frac{d\rho_{t,0}}{dt} + \epsilon \frac{d\rho_{t,1}}{dt} + \mathcal{O}(\epsilon^2) = \frac{Q_0 \rho_{t,0}}{\epsilon} + Q_0 \rho_{t,1} + Q_1 \rho_{t,0} + \epsilon Q_0 \rho_{t,2} + \epsilon Q_1 \rho_{t,1} + \mathcal{O}(\epsilon^2). \quad (\text{B.3})$$

Collecting terms of order $\mathcal{O}(\frac{1}{\epsilon})$ and $\mathcal{O}(1)$ with respect to parameter ϵ , we obtain equations

$$\begin{aligned} Q_0 \rho_{t,0} &= 0, \\ \frac{d\rho_{t,0}}{dt} &= Q_0 \rho_{t,1} + Q_1 \rho_{t,0}. \end{aligned} \quad (\text{B.4})$$

Since Q_0 is a block diagonal matrix of form (1.2), the first equation in (B.4) can be written as $Q_{0,i} \rho_{t,0}(i, \cdot) = 0$, $1 \leq i \leq m$. It follows from the irreducibility of each Markov chain \mathcal{C}_i that $\rho_{t,0}$ is constant on each subset X_i . And we can assume $\rho_{t,0}(x) = \bar{\rho}_t(i)$ for $x \in X_i$, where function $\bar{\rho}_t : \bar{X} \rightarrow \mathbb{R}$. Then the second equation of (B.4) can be written more explicitly as

$$\frac{d\bar{\rho}_t}{dt} = \sum_{x' \in X_i} Q_{0,i}(x, x') \rho_{t,1}(x') + \sum_{j \neq i} \sum_{y \in X_j} Q_1(x, y) (\bar{\rho}_t(j) - \bar{\rho}_t(i)), \quad (\text{B.5})$$

where $1 \leq i \leq m$ and $x \in X_i$. Now we multiply both sides of the above equation by $\pi_i(x)$ and sum up $x \in X_i$. Using $\pi_i^T Q_{0,i} = 0$ and the definition \bar{Q} in (1.3), we arrive at

$$\frac{d\bar{\rho}_t}{dt} = \bar{Q} \bar{\rho}_t. \quad (\text{B.6})$$

From expansion (B.2), the above reasoning indicates the convergence of ρ_t in (B.1) to $\bar{\rho}_t$ in (B.6). See Theorem 2.1 in Section 2.

The asymptotic behavior of Poincaré constant λ_ϵ , logarithmic Sobolev constant α_ϵ and modified logarithmic Sobolev constant γ_ϵ can be studied as well. Let f^ϵ be a function where the infimum in (1.5) is achieved. Then standard variation method implies

$$-\frac{Q + Q^*}{2} f^\epsilon = \lambda_\epsilon f^\epsilon, \quad (\text{B.7})$$

with $|f^\epsilon|_{L^2(\pi^\epsilon)} = 1$. Similarly, the minimizer of (1.6) satisfies

$$-\frac{Q+Q^*}{2}f^\epsilon = \alpha_\epsilon f^\epsilon \ln(f^\epsilon)^2, \quad (\text{B.8})$$

with $|f^\epsilon|_{L^2(\pi^\epsilon)} = 1$, while the minimizer of (1.10) satisfies

$$-Q^*f^\epsilon - f^\epsilon Q \ln f^\epsilon = \gamma_\epsilon f^\epsilon \ln f^\epsilon, \quad (\text{B.9})$$

with $\mathbf{E}_{\pi^\epsilon} f^\epsilon = 1$. For simplicity, we only consider the modified logarithmic Sobolev constant γ_ϵ using (B.9), since constants λ_ϵ and α_ϵ can be studied in a similar way. Assume we have the expansion

$$f^\epsilon = f_0 + \epsilon f_1 + \dots, \quad \gamma_\epsilon = \gamma_0 + \epsilon \gamma_1 + \dots. \quad (\text{B.10})$$

Substituting it into (B.9), we have

$$\begin{aligned} & -\frac{Q_0^* f_0}{\epsilon} - Q_1^* f_0 - Q_0^* f_1 - \frac{f_0 Q_0 \ln f_0}{\epsilon} - f_0 Q_1 \ln f_0 - f_1 Q_0 \ln f_0 - f_0 Q_0 \left(\frac{f_1}{f_0} \right) + \mathcal{O}(\epsilon) \\ & = \gamma_0 f_0 \ln f_0 + \mathcal{O}(\epsilon). \end{aligned}$$

Collecting terms of order $\mathcal{O}(\frac{1}{\epsilon})$ and $\mathcal{O}(1)$ with respect to parameter ϵ , we obtain equations

$$\begin{aligned} & Q_0^* f_0 + f_0 Q_0 \ln f_0 = 0, \\ & -Q_1^* f_0 - Q_0^* f_1 - f_0 Q_1 \ln f_0 - f_1 Q_0 \ln f_0 - f_0 Q_0 \left(\frac{f_1}{f_0} \right) = \gamma_0 f_0 \ln f_0. \end{aligned} \quad (\text{B.11})$$

Now for each $1 \leq i \leq m$, we multiply both sides of the first equation of (B.11) by $\pi_i(x)$ and sum up $x \in X_i$. Using $(Q_{0,i}^*)^T \pi_i = 0$, we can obtain

$$\mathcal{E}_i(f_0(i, \cdot), \ln f_0(i, \cdot)) = 0,$$

where \mathcal{E}_i is the Dirichlet form of Markov chain \mathcal{C}_i . From Lemma 2.7 of [5], we know

$$\mathcal{E}_i(f_0^{\frac{1}{2}}(i, \cdot), f_0^{\frac{1}{2}}(i, \cdot)) \leq \frac{1}{2} \mathcal{E}_i(f_0(i, \cdot), \ln f_0(i, \cdot)) = 0.$$

Since Markov chain \mathcal{C}_i is irreducible, we can deduce that f_0 is constant on each subset X_i , i.e. we have $f_0(x) = \bar{f}(i)$ when $x \in X_i$, where $\bar{f} : \bar{X} \rightarrow \mathbb{R}$. Now we multiply both sides of the second equation in (B.11) by $\pi_i(x)$ and sum up $x \in X_i$. Using the fact that $Q_0 \ln f_0 = 0$, $Q_{0,i}^T \pi_i = (Q_{0,i}^*)^T \pi_i = 0$, we can deduce that

$$-\bar{Q}^* \bar{f} - \bar{f} \bar{Q} \ln \bar{f} = \gamma_0 \bar{f} \ln \bar{f}, \quad (\text{B.12})$$

with $\mathbf{E}_w \bar{f} = 1$ (see Theorem 3.1). Comparing to (B.9), this equation shows that function \bar{f} is a minimizer of

$$\frac{\bar{\mathcal{E}}(\bar{f}, \ln \bar{f})}{\text{Ent}_w \bar{f}} \quad (\text{B.13})$$

and takes value γ_0 . Using the fact that $\bar{\gamma}$ is the infimum of (B.13) and the fact $\lim_{\epsilon \rightarrow 0} \gamma_\epsilon \leq \bar{\gamma}$ (see Theorem 3.2), we have

$$\bar{\gamma} \leq \gamma_0 = \lim_{\epsilon \rightarrow 0} \gamma_\epsilon \leq \bar{\gamma}.$$

Therefore we conclude that $\lim_{\epsilon \rightarrow 0} \gamma_\epsilon = \bar{\gamma}$.

REFERENCES

- [1] H. ABDI AND L. J. WILLIAMS, *Principal component analysis*, WIRs : Comp Stat, 2 (2010), pp. 433–459.
- [2] S. G. BOBKOV AND P. TETALI, *Modified logarithmic Sobolev inequalities in discrete settings*, J. Theoret. Probab., 19 (2006), pp. 289–336.
- [3] P. CAPUTO, P. DAI PRA, AND G. POSTA, *Convex entropy decay via the Bochner-Bakry-Émery approach*, Ann. Inst. H. Poincaré Probab. Statist., 45 (2009), pp. 734–753.
- [4] P. DEUFLHARD, W. HUISINGA, A. FISCHER, AND CH. SCHÜTTE, *Identification of almost invariant aggregates in reversible nearly uncoupled Markov chains*, Linear Algebra Appl., 315 (2000), pp. 39 – 59.
- [5] P. DIACONIS AND L. SALOFF-COSTE, *Logarithmic Sobolev inequalities for finite Markov chains*, Ann. Appl. Probab., 6 (1996), pp. 695–750.
- [6] W. E, D. LIU, AND E. VANDEN-EIJNDEN, *Nested stochastic simulation algorithms for chemical kinetic systems with multiple time scales*, J. Comput. Phys., 221 (2007), pp. 158–180.
- [7] M. ERBAR AND J. MAAS, *Ricci curvature of finite Markov chains via convexity of the entropy*, Arch. Ration. Mech. and Anal., 206 (2012), pp. 997–1038.
- [8] M. GIRVAN AND M. E. J. NEWMAN, *Community structure in social and biological networks*, Proc. Natl. Acad. Sci., 99 (2002), pp. 7821–7826.
- [9] L. GROSS, *Logarithmic Sobolev inequalities*, Amer. J. Math., 97 (1975), pp. 1061–1083.
- [10] A. GUIONNET AND B. ZEGARLINSKI, *Lectures on logarithmic Sobolev inequalities*, in Séminaire de probabilités XXXVI, vol. 1801 of Lecture Notes in Mathematics, 2003, pp. 1–134.
- [11] I. T. JOLLIFFE, *Principal component analysis*, Springer Series in Statistics, Springer, second ed., 2002.
- [12] M. LEDOUX, *Concentration of measure and logarithmic Sobolev inequalities*, in Séminaire de Probabilités XXXIII, vol. 1709 of Lecture Notes in Mathematics, Springer Berlin Heidelberg, 1999, pp. 120–216.
- [13] M. LEDOUX, *Spectral gap, logarithmic Sobolev constant, and geometric bounds*, in Surveys in Diff. Geom., Vol. IX, 219240, Int. Press, 2004, p. 2195409.
- [14] D. A. LEVIN, Y. PERES, AND E. L. WILMER, *Markov chains and mixing times*, Providence, R.I. American Mathematical Society, 2009. With a chapter on “coupling from the past” by J. G. Propp and D. B. Wilson.
- [15] Y. LIN AND S. T. YAU, *Ricci curvature and eigenvalue estimate on locally finite graphs.*, Math. Res. Lett., 17 (2010), pp. 343–356.
- [16] A. J. MAJDA, C. FRANZKE, AND B. KHOUIDER, *An applied mathematics perspective on stochastic modelling for climate*, Phil. Trans. R. Soc. A, 366 (2008), pp. 2429–2455.
- [17] S. MEYN AND R. L. TWEEDIE, *Markov Chains and Stochastic Stability*, Cambridge University Press, New York, NY, USA, 2nd ed., 2009.
- [18] M. E. J. NEWMAN, *Fast algorithm for detecting community structure in networks*, Phys. Rev. E, 69 (2004), p. 066133.
- [19] J. R. NORRIS, *Markov chains.*, Cambridge series in statistical and probabilistic mathematics, Cambridge University Press, 1998.
- [20] G. PAPANICOLAOU, A. BENSOUSSAN, AND J.L. LIONS, *Asymptotic Analysis for Periodic Structures*, Studies in Mathematics and its Applications, Elsevier Science, 1978.
- [21] G.A. PAVLIOTIS AND A. STUART, *Multiscale Methods: Averaging and Homogenization*, Texts in Applied Mathematics, Springer New York, 2008.
- [22] J. H. PRINZ, H. WU, M. SARICH, B. KELLER, M. SENNE, M. HELD, J. D. CHODERA, CH. SCHÜTTE, AND F. NOÉ, *Markov models of molecular kinetics: Generation and validation*, J. Chem. Phys., 134 (2011).
- [23] W. ZHANG, T. LI, AND P. ZHANG, *Numerical study for the nucleation of one-dimensional stochastic cahn-hilliard dynamics*, Commun. Math. Sci, (2011), pp. 1105–1132.